THE ELEMENTS OF GEOMETRY

COMPLETE EDITION

THE ELEMENTS OF GEOMETRY

IN

THEORY AND PRACTICE

BY

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AUTHOR OF "A MENSURATION FOR INDIAN SCHOOLS AND COLLEGES," ETC.
SOMETIME EXAMINER IN MATHEMATICS TO THE ALLAHABAD
AND PUNJAB UNIVERSITIES

COMPLETE EDITION

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PREFACE TO PARTS I., II. AND III.

THE aim of this book is to provide a course in the Elements of Geometry embodying those recent reforms in Geometrical Teaching that have been generally approved and adopted.

Parts I., II. and III. cover the ground of Euclid, Books I., II., III. and Book IV., 1-9 and 15. They also treat of Scales, Coordinates, Graphs, the Field Book, Pattern Drawing the Forms of Simple Solids and the Graphical Extraction of Arithmetical Square Root.

Each Part comprises (i) an Experimental Section, (ii) a Theoretical Section, (iii) a Practical Section. The Experimental Sections are introductory, and are intended to train the beginner in neatness, accuracy and the use of graduated ruler, dividers, protractor, set squares and compasses. Dividers are used from the beginning so as to avoid the error of parallax. No formal definitions are given, but the beginner is led to discover for himself the significance of terms and the properties of figures by sets of simple experiments. The Theoretical and Practical Sections are intended to be read concurrently. Terms are now formally defined and those Geometrical Truths that were *assumed* as the result of experiment in the Experimental Sections are now *proved* theoretically. By using Hypothetical Constructions, Theorems are rendered independent of Problems, while Problems are able to draw upon a wider range of Theorems for the proofs of their constructions.

In the sequence of Theorems and in the selection of Problems I have followed the Schedules sanctioned by the Senate of the

Cambridge University for the Previous Examination. By slight extensions, however, I have found it possible to embrace, within the limits of the subject-matter, the courses prescribed for the Oxford Responsions, the London Matriculation, the Oxford and Cambridge Locals and the Matriculation Examinations of the Indian Universities. The exercises have largely been drawn from past examination papers. Of the rest, many are original, but some have been taken from text-books on the subject, only, however, when they have been found to occur in two or more. Each set is graduated and, in Part I., is generally accompanied by an Examination Paper which is always retrospective, so that the pupil may not lose touch with what has gone before. All but the easiest exercises are marked with an asterisk and hints to their solutions are given at the end of the book, while many of the Questions for Examination are followed by their answers in brackets. The attention of the student is particularly drawn to the exercises printed in italics because they give results of importance.

I am much indebted to Mr. E. Sutcliffe, B.Sc., for his careful reading of the manuscript and proof sheets, and for many valuable suggestions.

A. E. PIERPOINT.

LUCKNOW, *December*, 1907.

PREFACE TO SECOND EDITION.

In this edition I have revised the text and made such extensions and alterations as I deemed desirable. At the same time I have added formal Propositions on the Construction of Regular Polygons in and about a circle and on the Concurrence of Straight Lines in a Triangle, so that Parts I., II., and III. may satisfy all the requirements of the Syllabus recently published for the School-leaving Examination of the University of Bombay.

A. E. PIERPOINT.

1928.

PREFACE

Part IV. treats of Ratio, Proportion, the Proportional Division of Straight Lines and Similar Triangles. It also treats of Maxima and Minima. The book will now be found to cover the Matriculation Syllabuses of all the English and Indian Universities.

A. E. PIERPOINT.

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Practical Section.

PART II.

Experimental Section.

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Theoretical Section.

PART IV.

Experimental Section.

Theoretical Section.

Practical Section.

Instruments and Apparatus Required.

Two black lead pencils—one H and the other HH—sharpened to a chisel edge.
Glass paper, for keeping a sharp edge on the lead pencils.
A flat ruler graduated in inches and tenths of an inch and also in centimetres and tenths of a centimetre (millimetres).
A semi-circular protractor.
A pair of pencil compasses.
A pair of dividers.
Two set-squares—one having angles of 90°, 45°, 45°, and the other having angles of 90°, 60°, 30°.
Scissors.
Tracing paper.
India-rubber.
Gummed paper.
Thin cardboard.
Tenth inch squared paper.

SUGGESTED ORDER FOR READING THE THEOREMS AND PROBLEMS.

PART I.

Theorems 1 to 15.
Problems 1 to 6.
Theorems 16 to 25,
and Additional Theorems I. to IV.
Problems 7 to 12.

PART II.

Theorems 26 to 33.
Problem 13.
Theorems 34 to 39.
Problems 14, 15.
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Problems 16 to 23,
and Additional Problems I., II.

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Theorems A to F.
Theorems 49 to 53.
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PART I.

EXPERIMENTAL SECTION.

DRAWING AND MEASUREMENT OF STRAIGHT LINES.

Exp. 1. Draw straight lines of the following lengths, using your ruler: 2 in., 3·5 in., 5·4 in., 13 cm., 10·7 cm., 90 mm., 7·4 in., 16·7 cm.

Rule from left to right, and be careful that the lines are the same thickness throughout.

Write the length of each line along it as in Fig. 1.

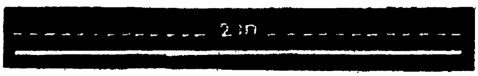


FIG. 1.

If the line to be drawn is longer than the whole length of the ruler, draw it in parts, but take care that the parts run on so as not to show where they join.

Exp. 2. Draw half a dozen straight lines of different lengths with your set-square. Measure their lengths with your dividers and graduated ruler (i) in inches, (ii) in centimetres.

Write the length of each line along it as in Exp. 1.

In taking lengths with your dividers (if they have not a screw adjustment) always open them wide to begin with and then close them to the given length by pressing the legs together. Press lightly on the points so as not to scratch the ruler or puncture the paper.

In all cases, if the length you are measuring does not end exactly at one of the divisions marked on the scale, *guess* its measurement to the nearest hundredth of an inch or hundredth of a centimetre (*i.e.*, tenth of a millimetre) according as you are using inches or centimetres. For example:—

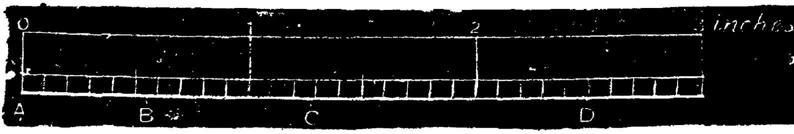


FIG. 2.

AB = 0·55 in.
AC = 1·27 in.
AD = 2·49 in.

If the line you are measuring is longer than the whole length of your ruler or the span of your dividers, measure parts of it separately and then add the results.

Exp. 3. Draw four straight lines of different lengths with your set-square, number them, and *guess* their lengths (i) in inches, (ii) in centimetres. Test your guesses by measuring with your dividers and ruler and tabulate your answers thus:—

Line	Guessed Length	True Length
1		
2		
3		
4		

Exp. 4. Guess the number of (i) inches, (ii) centimetres in the following, and test your guesses by measuring with your dividers and ruler: the length of this page, the length of your pencil, the breadth of your ruler, the thickness of this book. Tabulate your answers as in Exp. 3.

Exp. 5. Draw *from memory*, as accurately as you can, with your set-square, straight lines of the following lengths: 1 in., 1 cm., 2·5 in., 3·2 cm., 5 in., 11 cm. Test your guesses by measuring with your dividers and ruler. Write the supposed length and true length along each.

Exp. 6. Draw a straight line 4·3 in. long, using your dividers, and measure its length in centimetres. Hence find the number of centimetres in an inch correct to the first decimal place. Do the same for a straight line 3·8 inches long, and see that in each case you get the same number of centimetres to the inch.

ADDITION AND SUBTRACTION AND DIVISION OF LENGTHS.

Exp. 7. Measure AB, BC, CD in inches, using your dividers. Add the results, and verify by measuring the whole line AD.

FIG. 3.

Tabulate thus :—

Length of AB =		inches.
,, ,, BC =		,,
,, ,, CD =		,,
,, ,, AD =		,,

Exp. 8. Repeat Exp. 7, using centimetres instead of inches.

Exp. 9. Measure AB and BC in inches, using your dividers. Subtract the results and verify by measuring AC.

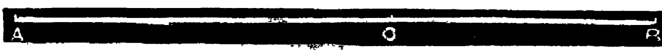


FIG. 4.

Tabulate thus :—

Length of AB =		inches.
,, ,, BC =		,,
,, ,, AC =		,,

Exp. 10. Repeat Exp. 9, using millimetres instead of inches.

Exp. 11. Draw a straight line 5·3 in. long, using your dividers, and from it mark off parts equal to 1·7 in., 0·9 in. and 1·1 in. respectively. Show both by calculation and measurement that what is left equals 1·6 in.

Exp. 12. Draw half a dozen straight lines of different lengths with your set-square. Measure their lengths with your dividers and ruler. Halve each measurement and so arrive at their middle points. Make a mark to show the middle point of each.

If halving a measurement yields a 3rd decimal place you may ignore it.

Exp. 13. Draw straight lines of the following lengths using your dividers, and find their middle points *by calculation* as in Exp. 12 :—5 in., 13 cm., 7·3 in., 19·5 cm., 6·9 in., 91 mm.

Exp. 14. Draw half a dozen straight lines of different lengths with your set-square. Let each line measure at least 5 in. Mark what you *guess* to be their middle points and then find their true middle points by calculation as in Exp. 12.

Exp. 15. Draw straight lines of the following lengths and divide them into (i) thirds, (ii) fifths, *by calculation:* 6 in., 4·5 in., 15 cm., 13·5 cm.

Exp. 16. Draw three straight lines of different lengths with your set-square. Let each line measure at least 5 in. Number them and divide them into (i) thirds, (ii) fifths, by *guess-work.* Test your guesses by measurement, and make a table to show your error thus:—

Line 1.

Supposed thirds, 1·67 in., 1·70 in., 1·76 in.
" fifths,

DRAWING AND MEASUREMENT OF ANGLES.

Exp. 17. Draw two straight lines OA and OB from the same point O in different directions, thus:—

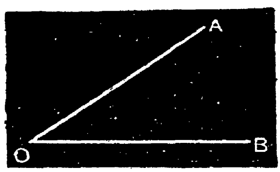


FIG. 5.

OA and OB are said to form the angle AOB (or BOA, or simply O) with one another, and they themselves are called the arms of the angle. The point O is called its vertex.

We shall see later that the size of an angle in no way depends upon the lengths of its arms.

If the arms of an angle are square with one another, thus:—

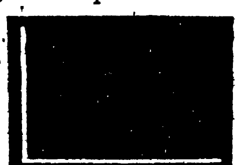


FIG. 6.

the angle is called a right angle.

Exp. 18. Draw a right angle using your *set-square* and *ruler*, and let each of its arms measure one inch.

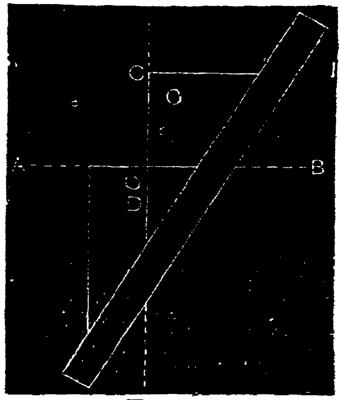


FIG. 7.

Fig. 7 will show you how to do this. Rule **AB**. Place the set-square and ruler in position, slide the set-square from one position to the other along the ruler, and then rule **CD**.

Exp. 19. Draw a right angle, using your *protractor*, and let each

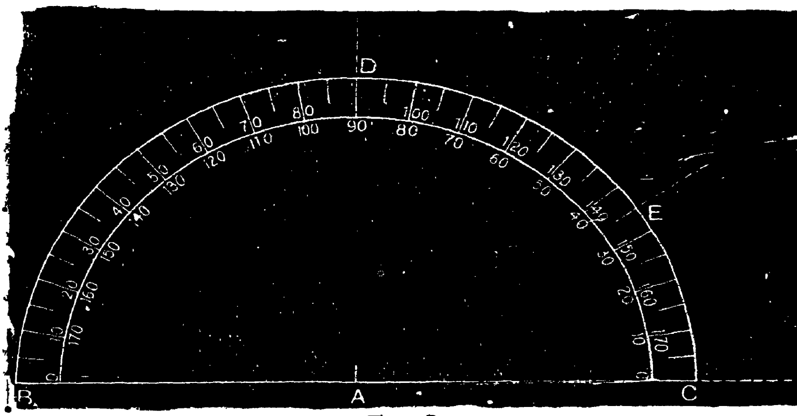


FIG. 8.

of its arms measure 4·2 cm.

Fig. 8 will show you how to do this. Notice that the protractor is placed with its base along one of the arms of the angle, and the other arm is a straight line drawn from **A**, the centre of the protractor, so as to pass underneath **D**, the graduation marked 90. A pin prick at **A** and another at **D** will help you to draw the second arm of the angle accurately.

Exp. 20. Draw a right angle using your protractor. Let its arms be any length you like. Cut it out of the paper with your scissors thus :—

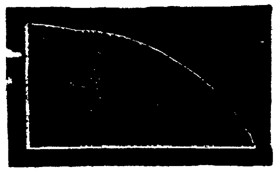


FIG. 9.

Now fold it so that one arm lies along the other and open it out again. The crease divides the right angle into two equal angles thus :—

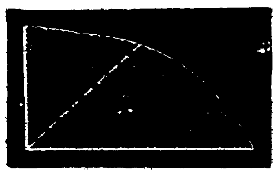


FIG. 10.

Go on cutting and folding until you have divided the right angle into four equal angles or fourths of a right angle.

The ninetieth part of a right angle is called a degree. Hence there are 45 degrees (written 45°) in half a right angle, and $22\frac{1}{2}$° in a quarter of a right angle. How many degrees are there in

(1) An eighth of a right angle? (Ans. $11\frac{1}{4}$.)
(2) Two right angles? (Ans. 180.)

Exp. 21. Draw an angle of 35° using your protractor.

Fig 8 will show you how to do this. Notice that the protractor is placed with its base along one of the arms of the angle and then the other arm is a straight line drawn from A, the centre of the protractor, so as to pass underneath E, the graduation 35.

Notice, also, that two sets of numbers are given on the protractor—one set giving the degrees in angles that have an arm lying along AC, and the other set giving the degrees in angles that have an arm lying along AB. Be careful to use the right set

Exp. 22. Draw the following angles with your protractor: 25°, 72°, 9°, 70°, 173°, 113°.

Write its measurement inside each angle thus:—

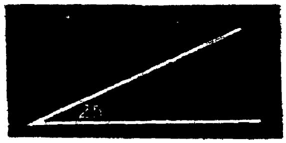


FIG. 11.

Exp. 23. Draw half a dozen angles of different sizes with your ruler and then measure them with your protractor.

Write its measurement inside each angle.

In all cases, if the angle you are measuring does not end exactly at one of the divisions marked on the protractor, *guess* its measurement to the nearest half degree.

Exp. 24. Measure the angles of your set-squares. (Ans. 30°, 60°, 90°, 45°, 45°, 90°.)

Exp. 25. Draw three angles of different sizes with your ruler, number them and *guess* how many degrees there are in each. Test your guesses by measuring with your protractor and tabulate your answers thus:—

Angle	Guessed Degrees	True Degrees
1		
2		
3		

Exp. 26. Draw *from memory*, as accurately as you can, with your ruler, angles of the following sizes: 45°, 30°, 60°, 10°, 80°, 135°.

Test your guesses by measuring with your protractor and write the supposed size and the true size inside each.

ADDITION AND SUBTRACTION AND DIVISION OF ANGLES.

Exp. 27. Measure the angles AOB, BOC, COD, DOE with your protractor, add the results and verify by measuring the whole angle AOE.

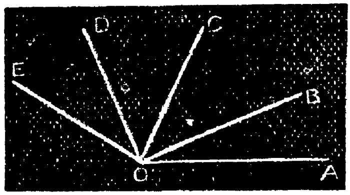


FIG. 12.

Tabulate thus :—

Angle AOB =		degrees.
„ BOC =		„
„ COD =		„
„ DOE =		„
„ AOE =		„

Exp. 28. Measure the angles AOC and AOB with your protractor, subtract the results and verify by measuring the angle BOC.

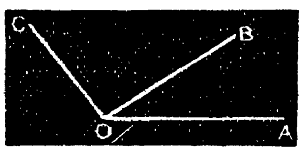


FIG. 13.

Tabulate thus :—

Angle AOC =
„ AOB =

„ BOC =

Exp. 29. Draw the angle 113° and from it take away parts of 19°, 37°, 54° respectively, using your protractor. Show, both by calculation and measurement, that what is left measures 3°.

Exp. 30. Draw half a dozen angles of different sizes with your ruler. Measure them with your protractor. Divide each measurement by two and so arrive at their halves. Draw the line of bisection in each case, that is, the line that divides the angle into two equal parts.

Exp. 31. Draw angles of the following sizes with your protractor and find their lines of bisection *by calculation*, as in Exp. 30: 38°, 86°, 112°, 164°, 65°, 103°.

Exp. 32. Draw half a dozen angles of different sizes with your ruler—three less than a right angle and three greater than a right angle. Draw what you *guess* to be their lines of bisection, and then find their true lines of bisection by calculation as in Exp. 30.

Exp. 33. Draw the following angles and divide them into (1) thirds, (2) fifths, *by calculation:* 60°, 90°, 30°, 120°, 105°, 180°.

Exp. 34. Draw three angles of different sizes with your ruler—one less than a right angle and two greater than a right angle. Number them and divide them into (1) thirds, (2) fifths, by *guesswork*. Test your guesses by measurement and make a table to show your error thus:—

Angle 1:—

Supposed thirds, $25\frac{1}{2}$°, $25\frac{1}{2}$°, 24°.
,, fifths,

In the pages that follow we shall often find it convenient to use the letters of the Greek Alphabet, especially for denoting angles. These are they:—

α	pronounced	alpha	ν	pronounced	nu
β	,,	beta	ξ	,,	xi
γ	,,	gamma	o	,,	omicron
δ	,,	delta	π	,,	pi
ϵ	,,	epsilon	ρ	,,	rho
ζ	,,	zēta	σ	,,	sigma
η	,,	ēta	τ	,,	tau
θ	,,	thēta	υ	,,	upsilon
ι	,,	iōta	ϕ	,,	phi
κ	,,	kappa	χ	,,	chi
λ	,,	lambda	ψ	,,	psi
μ	,,	mu	ω	,,	omĕga

ANGLES AT A POINT.

Exp. 35. Draw one straight line standing on another straight line thus:—

FIG. 14.

Measure the two angles α and β so formed with your protractor, add their measures and tabulate your results:—

α = degrees.
β = ,,

$\alpha + \beta$ =

Exp. 36. Repeat Exp. 35 with half a dozen pairs of straight lines and say what conclusion you always come to in regard to the sum of each pair of angles.

From these experiments we are led to conclude :—

If a straight line stands on another straight line, the sum of the two angles so formed is equal to two right angles.

Learn this by heart.

Exp. 37. Draw two angles whose measures when added together make up 180° thus :—

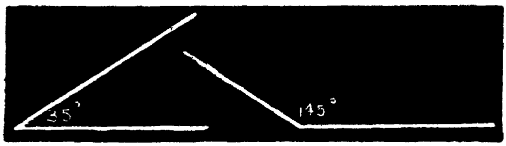


FIG. 15.

Cut them out with your scissors and fit them together so that the vertex and an arm of one fall on the vertex and an arm of the other but each angle falls outside the other thus :—

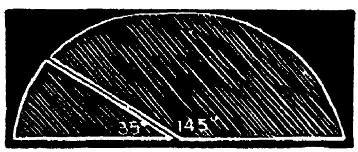


FIG. 16.

Lay your ruler along the other arms of the angles and notice that they lie in one and the same straight line.

Exp. 38. Repeat Exp. 37 with half a dozen pairs of angles and say what conclusion you always come to in regard to the two arms of the angles that do not fall on one another.

From these experiments we are led to conclude :—

If the sum of two adjacent angles is equal to two right angles the exterior arms of the angles are in the same straight line.

Learn this by heart. (The angles are said to be *adjacent* because they *lie next to* one another.)

Exp. 39. Draw one straight line cutting another straight line thus :—

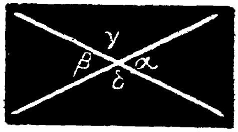


FIG. 17.

Of the four angles thus formed, α, β, γ, δ, measure (1) the opposite pair α and β, (2) the opposite pair γ and δ. Tabulate your results.

$\alpha =$	degrees.	$\gamma =$	degrees.
$\beta =$	"	$\delta =$	"

Exp. 40. Repeat Exp. 39 with half a dozen pairs of straight lines cutting one another, and say what conclusion you always come to in regard to the sizes of opposite angles.

Exp. 41. Illustrate the conclusion you arrived at in Exp. 40 by the following experiment: Draw two intersecting straight lines, cut out one of the angles so formed as in Exp. 37, and show that it can be made to fit exactly over its opposite angle.

From these experiments we are led to conclude :—

If two straight lines intersect, the vertically opposite angles are equal.

Learn this by heart. (The angles are said to be *vertically* opposite because they have the same vertex.)

PARALLEL STRAIGHT LINES.

Exp. 42. Move your set-square along your ruler so as to occupy two positions, as in Fig. 18.

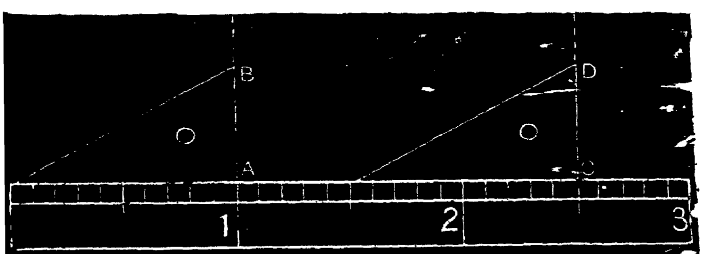


FIG. 18.

Rule the straight lines AB and CD. These straight lines will be parallel, and they are 1¼ inches apart.

Exp. 43. Draw two straight lines parallel to one another and $1\frac{7}{16}$ inches apart, using your ruler and set-square.

Exp. 44. Draw a straight line parallel to a given straight line and distant $2\frac{1}{4}$ inches from it.

In this case AB in the first position of the set-square (Fig. 18) must lie along the given straight line.

Exp. 45. Through a given point draw a straight line parallel to a given straight line.

In this case AB in the first position of the set-square (Fig. 18) must lie along the given straight line, and CD in the second position must pass through the given point.

Exp. 46. Draw two straight lines parallel to one another and one inch apart, *using only your pencil.* Test your guess with your ruler and set-square.

Exp. 47. Draw a straight line cutting two parallel straight lines thus :—

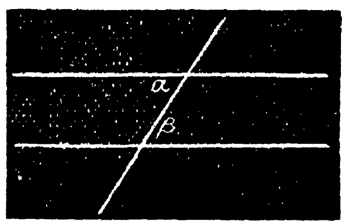


FIG. 19.

Measure the "alternate" angles α and β and make a note of their measurements.

Exp. 48. Repeat Exp. 47 with half a dozen different positions of the cutting line, and say what conclusion you always come to in regard to the alternate angles.

Exp. 49. Illustrate the conclusion you arrived at in Exp. 48 by cutting out either of a pair of alternate angles and fitting it over the other.

Exp. 50. Draw a straight line cutting two parallel straight lines thus :—

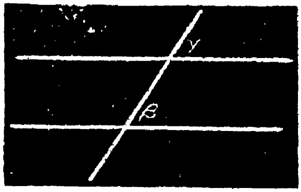


Fig. 20.

Measure the "corresponding" angles β and γ, and make a note of their measurements.

Exp. 51. Repeat Exp. 50 with half a dozen different positions of the cutting line, and say what conclusion you always come to in regard to the corresponding angles.

Exp. 52. Illustrate the conclusion you arrived at in Exp. 51 by cutting out either of a pair of corresponding angles and fitting it over the other.

Exp. 53. Draw a straight line cutting two parallel straight lines thus :—

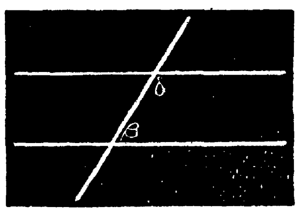


Fig. 21.

Measure β and δ the two "interior" angles on the same side of the cutting line, and write down the number of degrees in their measurements added together.

$$\begin{array}{rl} \beta = & \quad \text{degrees.} \\ \delta = & \quad \text{,,} \\ \hline \beta + \delta = & \quad \text{,,} \end{array}$$

Exp. 54. Repeat Exp. 53 with half a dozen different positions of the cutting line, and say what conclusion you always come to in regard to the interior angles on the same side of the cutting line.

Exp. 55. Illustrate the conclusion you arrived at in Exp. 54 by cutting out a pair of interior angles and fitting them together so that the vertex and an arm of one may fall upon the vertex and an arm of the other, but each angle may fall outside the other thus :—

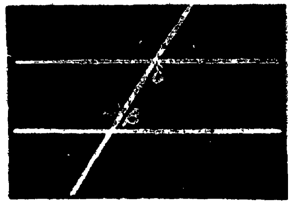


FIG. 22.

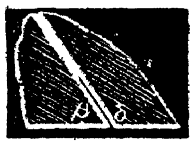


FIG. 23

From experiments 47 to 55 we are led to conclude :—

If a straight line cuts two parallel straight lines—

(1) *Alternate angles are equal.*

(2) *Corresponding angles are equal.*

(3) *The interior angles on the same side of the cutting line are together equal to two right angles.*

Learn this by heart.

Exp. 56. Make three figures of a straight line cutting two other straight lines, but in the first figure make a pair of alternate angles equal to one another, in the second figure make a pair of corresponding angles equal to one another, and in the third figure let the two interior angles on one side of the cutting line be together equal to 180°. Show by means of your ruler and set-square that in all three cases two of the straight lines are parallel to one another.

From these experiments we are led to conclude :—

When a straight line cuts two other straight lines, if

(1) *a pair of alternate angles are equal, or*

(2) *a pair of corresponding angles are equal, or*

(3) *a pair of interior angles on the same side of the cutting line are together equal to two right angles,*

then the two straight lines are parallel.

Learn this by heart.

Exp. 57. Draw two straight lines parallel to the same straight line (see Exp. 44), and show by means of your ruler and set-square that they are parallel to one another.

From this experiment we are led to conclude :—

Straight lines which are parallel to the same straight line are parallel to one another.

Learn this by heart.

ANGLES OF A TRIANGLE.

Exp. 58. Draw half a dozen figures of different shapes, but each enclosed by only three straight lines. Call them triangles.

Exp. 59. Draw a triangle. Let a, β and γ be its three "interior" angles. Measure a, β and γ. Add their measurements, and note the result.

$$
\begin{array}{rll}
a = & & \text{degrees.} \\
\beta = & & \text{"} \\
\gamma = & & \text{"} \\
\hline
a + \beta + \gamma = & & \text{"}
\end{array}
$$

Exp. 60. Repeat Exp. 59 with half a dozen different triangles, and say what conclusion you always come to in regard to the sum of the interior angles of each triangle.

Exp. 61. Illustrate the conclusion you arrived at in Exp. 60 by the following experiment : Draw a triangle, cut out the three interior angles, and fit them together as in Fig. 24.

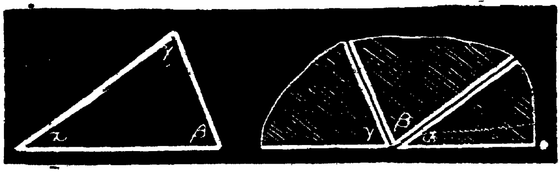


FIG. 24.

From these experiments we are led to conclude :—

The sum of the angles of a triangle is equal to two right angles.

Learn this by heart.

Exp. 62. Draw a triangle and produce one of its sides thus :—

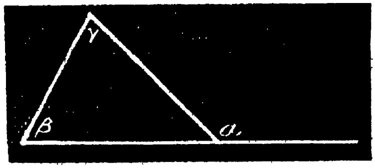


FIG. 25.

Measure the "exterior" angle α and the two interior and opposite angles β and γ. Compare the number of degrees in α with the *sum* of the numbers of degrees in β and γ.

degrees. $\beta =$ degrees.

$\gamma =$,,

$\beta + \gamma =$,,

Exp. 63. Repeat Exp. 62 with half a dozen triangles and say what conclusion you always come to when you compare the number of degrees in an exterior angle with the sum of the numbers of degrees in the two interior and opposite angles.

Exp. 64. Illustrate the conclusion you arrived at in Exp. 63 by cutting out and "superposing" (see Exp. 41) over an exterior angle the two interior and opposite angles of any triangle.

From these experiments we are led to conclude :—

If one side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.

Learn this by heart.

ANGLES OF ANY CONVEX POLYGON.

Exp. 65. Draw half a dozen figures of different shapes but each enclosed by four or more straight lines. Call them polygons.

A polygon is said to be convex when each of its interior angles is less than two right angles, or, in other words, when all its corners point outwards. A in Fig. 26 is a convex polygon. B in Fig. 26 is not.

FIG. 26.

Exp. 66. Draw a convex polygon. Produce its sides in order thus.—

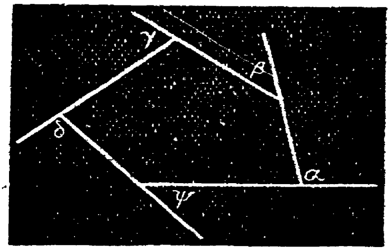


FIG. 27.

Let α, β, γ, δ, ψ be the exterior angles so formed. Measure α, β, γ, δ, ψ. Add their measurements and note the result:—

$\alpha =$		degrees.
$\beta =$		"
$\gamma =$		"
$\delta =$		"
$\psi =$		"
$\alpha + \beta + \gamma + \delta + \psi =$		"

Exp. 67. Repeat Exp. 66 with half a dozen convex polygons, varying the shape and number of sides, and say what conclusion you always come to in regard to the sum of the exterior angles of each polygon formed by producing its sides in the same order.

Exp. 68. Illustrate the conclusion you arrived at in Exp. 67 by the following experiment: Draw a convex polygon; produce its sides in order; cut out the exterior angles so formed, and fit them together so that their vertices meet and they form a perfect mosaic (see Fig. 24).

From these experiments we are led to conclude:—

If the sides of a convex polygon are produced in order, the sum of the angles so formed is equal to four right angles.

Learn this by heart.

Exp. 69. Draw a convex polygon and show by measurement that *the sum of its interior angles together with four right angles is equal to twice as many right angles as the polygon has sides.*

Learn this by heart.

CONSTRUCTION AND COMPARISON OF SPECIAL TRIANGLES

Exp. 70. Make a triangle ABC having the angle ABC = 75°, BA = 2·7 in. and BC = 1·9 in. Measure AC and the angles BAC

2

and BCA. (Ans. AC = 2·87 in.; angle BAC = 39½°; angle BCA = 65½°.)

To make this triangle begin by drawing a line 2·7 in. in length; at one extremity draw a line making with it an angle of 75° and from this line cut off a length equal to 1·9 in.

When making a figure to given measurements it is always well to begin by making a rough sketch of the figure in which the given measurements are written.

Exp. 71. Make triangles having the following measurements:—

(i)	Angle ABC = 90°.	BA = 1·2 in.	BC = 1·2 in.
(ii)	,, ,, = 60°.	,, = 1·5 in.	,, = 1·5 in.
(iii)	,, ,, = 25°.	,, = 1·4 in.	,, = 1·1 in.
(iv)	,, ,, = 136°.	,, = 2·8 cm.	,, = 3·1 cm.
(v)	,, ,, = 15½°.	,, = 4·3 cm.	,, = 3·9 cm.
(vi)	,, ,, = 175½°.	,, = 1·25 in.	,, = 1·47 in.

Exp. 72. Make a triangle ABC having the angle ABC = 63°, BA = 2·5 in. and BC = 1·8 in. Make another triangle ABC having these same measurements. Measure the remaining sides and angles of the two triangles and show that the two triangles are equal to one another in all respects.

Exp. 73. Show that the two triangles in Exp. 72 are equal in all respects by cutting one out and fitting it exactly over the other.

From these experiments we are led to conclude:—

> *If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles contained by those sides equal, the triangles are congruent, that is, equal in all respects.*

Learn this by heart.

Exp. 74. Make a triangle ABC having BC = 4·4 cm., the angle ABC = 63° and the angle ACB = 54°. Measure AB, AC and the angle BAC. (Ans. AB = 4 cm.; AC = 4·4 cm.; angle BAC = 63°.)

To make this triangle begin by making BC = 4·4 cm. and then set off angles at B and C equal to 63° and 54° respectively.

Exp. 75. Make triangles having the following measurements:—

(i)	BC = 1·5 in.	Angle ABC = 60°.	Angle ACB = 60°.
(ii)	,, = 1·25 in.	,, ,, = 90°.	,, ,, = 45°.
(iii)	,, = 3·8 cm.	,, ,, = 30°.	,, ,, = 60°.
(iv)	,, = 1·75 in.	,, ,, = 22°.	,, ,, = 103°
(v)	,, = 4·55 cm.	,, ,, = 54½°.	,, ,, = 97°.
(vi)	,, = 39·5 mm.	,, ,, = 118½°.	,, ,, = 31½°.

Exp. 76. Make a triangle ABC having BC = 2·3 in., the angle ABC = 28° and the angle ACB = 133°. Make another triangle ABC having these same measurements. Measure the remaining sides and angles of the two triangles and show that the two triangles are congruent.

Exp. 77. Show that the two triangles in Exp. 76 are congruent by cutting one out and fitting it exactly over the other.

Exp. 78. Make a triangle ABC having AB = 1·2 in., the angle ABC = 63° and the angle ACB = 75°. Measure BC, AC and the angle BAC. (Ans. BC=0·83 in.; AC=1·1 in.; angle BAC=42°.)

Since the sum of the angles of a triangle is equal to two right angles (see Exp. 61) we know that the angle CAB of the required triangle ABC is equal to 180° minus (63° + 75°), that is 42°. Hence the triangle may be described in the same way as the triangles of Exp. 75.

Exp. 79. Make triangles having the following measurements:—

(i)	BC = 1·4 in.	Angle BCA = 30°.	Angle BAC = 60°.
(ii)	„ = 1·05 in.	„ „ = 45°.	„ „ = 90°.
(iii).	„ = 3·6 cm.	„ „ = 43°.	„ „ = 72°.
(iv)	„ = 1·95 in.	„ „ = 28°.	„ „ = 17°.
(v)	„ = 42·5 mm.	„ „ = 31°.	„ „ = 109°.
(vi)	„ = 37·5 mm.	„ „ = 28½°.	„ „ = 22½°.

Exp. 80. Make a triangle ABC having BC = 3·2 cm., the angle BCA = 57° and the angle BAC = 71°. Make another triangle ABC having these same measurements. Measure the remaining sides and angles of the two triangles and show that the two triangles are congruent.

Exp. 81. Show that the two triangles in Exp. 80 are congruent by cutting one out and fitting it exactly over the other.

From experiments 74 to 81 we are led to conclude:—

If two triangles have two angles of the one equal to two angles of the other, each to each, and also one side of the one equal to the corresponding[1] *side of the other, the triangles are congruent.*

Learn this by heart.

ISOSCELES TRIANGLES.

Exp. 82. Make a triangle ABC having AB and AC each equal to 1·7 in., and BC of any length whatever. Measure the angles ABC and ACB and compare their measurements.

Exp. 83. Make half a dozen triangles of different shapes but each triangle having two of its sides equal to one another. Call them isosceles triangles. Compare the measurements of the two angles in each triangle that lie opposite to the equal sides and say what conclusion you always come to.

From these experiments we are led to conclude:—

If two sides of a triangle are equal, the angles opposite to these sides are equal.

Learn this by heart.

Exp. 84. Make a triangle ABC having the angle ABC and the angle ACB each equal to 63°, and BC of any length whatever. Measure AB and AC and compare their measurements.

[1] "Corresponding" sides are opposite to the equal angles one in each triangle

Exp. 85. Make half a dozen triangles of different shapes but each triangle having two of its angles equal to one another. Compare the measurements of the two sides in each triangle that lie opposite to the equal angles and say what conclusion you always come to.

From these experiments we are led to conclude :—

If two angles of a triangle are equal, the sides opposite to these angles are equal.

Learn this by heart.

CONSTRUCTION AND COMPARISON OF SPECIAL TRIANGLES.

Exp. 86. Describe a circle with your compasses and notice that the boundary of the circle, which is called its circumference, is always at the same distance from a point within the circle which is called its centre. This same distance is called the radius of the circle.

Hold your compasses by the head and press lightly on the needle point so as not to puncture the paper.

Exp. 87. Draw circles with your compasses having the following radii : 1 in., ¾ in., 2 cm., 1·8 cm., 0·9 in., 23 mm.

In taking lengths with your compasses, open them wide to begin with and then close them to the given length by pressing the legs together.

Exp. 88. Find half a dozen different points all distant 1¼ in. from a given point A, using your compasses.

Exp. 89. Draw a straight line AB = 1 in. Find a point C distant 1 in. from A and 1 in. from B. Show that C may occupy two positions.

Evidently C lies where the two circles cut one another that are described with centres A and B respectively and of radius = 1 in. See Fig. 28.

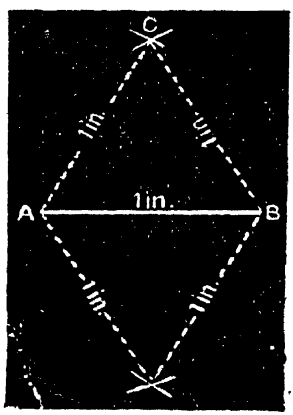


FIG. 28.

Exp. 90. Make a triangle whose three sides each measure $\frac{7}{10}$ of an inch.

Exp. 91. Draw a straight line AB = 4·3 cm. Find a point C distant 3·4 cm. from A and 2·5 cm. from B. Show that C may occupy two positions (see Fig. 29).

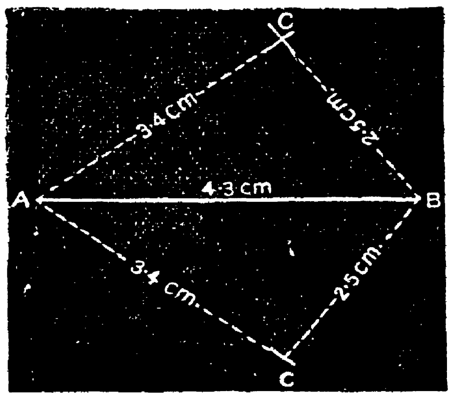


FIG. 29.

Exp. 92. Make triangles having the following measurements:—

(i)	AB = 1·2 in.	BC = 1 2 in.	AC = 1·2 in.
(ii)	,, = 1·5 in.	,, = 1·5 in.	,, = 0·8 in.
(iii)	,, = 1·3 in.	,, = 1·1 in.	,, = 0·9 in.
(iv)	,, = 2·8 cm.	,, = 3·1 cm.	,, = 2·4 cm.
(v)	,, = 1·25 in.	,, = 1·37 in.	,, = 1 12 in.
(vi)	,, = 27 mm.	,, = 32 mm.	,, = 26·5 mm.

Exp. 93. Try to make a triangle ABC having AB = 3 4 cm, BC = 1·3 cm., and AC = 1·8 cm. Why is the construction impossible?

From this experiment we are led to conclude:—

Any two sides of a triangle are together greater than the third side.

Learn this by heart.

Exp. 94. Make a triangle ABC having AB = 0·85 in., BC = 1·7 in., AC = 1·9 in. Make another triangle ABC having these same measurements. Measure the angles of the two triangles and show that the two triangles are congruent.

Exp. 95. Show that the two triangles in Exp. 94 are congruent by cutting one out and fitting it exactly over the other.

From these experiments we are led to conclude :—

If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles are congruent.

Learn this by heart.

Exp. 96. Make a triangle having one of its angles a right angle. Call it a right-angled triangle and call the side opposite the right angle its hypotenuse.

Exp. 97. Make a right-angled triangle ABC in which the angle ACB is the right angle, the hypotenuse AB = 3·4 cm. and AC = 1·5 cm. Measure BC, the angle ABC and the angle BAC. (Ans. BC = 3·05 cm., angle ABC = 26°, angle BAC = 64°.)

To do this, begin by making the right angle ACB; measure off CA = 1·5 cm., and then with centre A and radius = 3·4 cm. make an arc of a circle cutting CB in B (see Fig. 30).

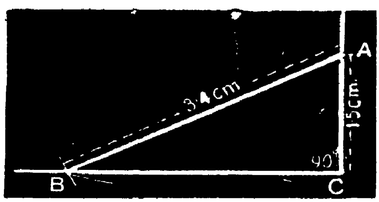


Fig. 30.

Exp. 98. Make the right-angled triangles having the following measurements :—

(i)	Hypotenuse AB	= 1·5 in.	AC = ·8 in.
(ii)	,, ,,	= 1·15 in.	,, = ·75 in.
(iii)	,, ,,	= 3·9 cm.	,, = 2·4 cm.
(iv)	,, ,,	= 37·5 mm.	,, = 22 mm.

Exp. 99. Make a right-angled triangle ABC in which the angle ACB is a right angle, the hypotenuse AB = 1·6 in. and AC = 1·1 in. Make another right-angled triangle ABC having these same measurements. Measure the remaining sides and angles of the two triangles, and show that the two triangles are congruent.

Exp. 100. Show that the two triangles in Exp. 99 are congruent by cutting one out and fitting it exactly over the other.

From these experiments we are led to conclude :—

If two right-angled triangles have their hypotenuses equal and one side of the one equal to one side of the other, the triangles are congruent.

Learn this by heart.

INEQUALITIES IN TRIANGLES.

Exp. 101. Make a triangle and measure two of its sides. Then measure the angles that are opposite to these sides. Tabulate your measurements. You will find that the greater side has the greater angle opposite to it and the greater angle has the greater side opposite to it.

Exp. 102. Repeat Exp. 101 with half a dozen triangles of different shapes, and learn by heart the conclusions that you always come to :—

(i) *If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.*

(ii) *If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.*

Exp. 103. Make two triangles having two sides of the one equal to two sides of the other, each to each, but not equal in all respects. Measure the contained angle and third side of each triangle. Tabulate your measurements. You will find that the triangle which has the greater contained angle has the greater third side, and the triangle which has the greater third side has the greater contained angle.

Exp. 104. Repeat Exp. 103 with half a dozen pairs of triangles, and learn by heart the conclusions that you always come to :—

(i) *If two triangles have two sides of the one equal to two sides of the other, each to each, and the contained angles unequal, the triangle which has the greater contained angle has the greater third side.*

(ii) *If two triangles have two sides of the one equal to two sides of the other, each to each, and the third sides unequal, the triangle which has the greater third side has the greater contained angle.*

Exp. 105. Illustrate by drawing a figure and by actual measurement the following truth :—

Of all the straight lines that can be drawn to a given straight line from a given point outside it, the perpendicular (or upright line) is the shortest.

Learn this by heart.

PARALLELOGRAMS.

Exp. 106. Draw half a dozen four-sided figures of different shapes, but make the opposite sides of each figure parallel to one another. Call them parallelograms.

The diagonals of a parallelogram are the straight lines joining opposite corners.

Exp. 107. Illustrate by drawing a figure and by actual measurement the following truths:—

The opposite sides and angles of a parallelogram are equal and the diagonals bisect one another.

Learn this by heart.

Exp. 108. Illustrate by drawing a figure, and by cutting out and superposing, the following truth:—

The diagonals of a parallelogram divide it into two equal parts.

Learn this by heart.

Exp. 109. Illustrate by drawing a figure, and by actual measurement, the following truth:—

If there are three or more parallel straight lines and the intercepts made by them on any straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are also equal.

Learn this by heart.

In drawing the figure for this experiment begin by taking points A, B, C . . . on any straight line such that "intercept" AB = "intercept" BC = . . .; then through A, B, C . . . draw straight lines parallel to one another and draw any other straight line to cut the parallels.

SOME SIMPLE SOLIDS.

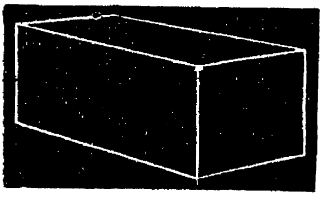


Fig. 31.

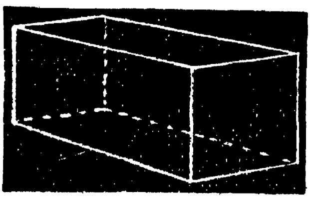


Fig. 32.

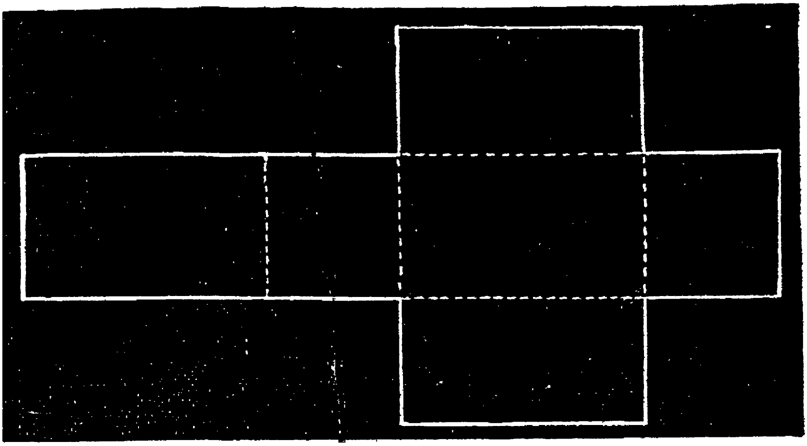


FIG. 33.

Figs. 31, 32, 33 represent a solid which we shall call a rectangular block or cuboid.

Exp. 110. Copy Fig. 33 on thin cardboard—tracing paper may be used with advantage. Cut out this copy and form a model of a rectangular block by bending along the dotted lines and joining the edges with gummed paper. Fig. 33 is called the "net" of a rectangular block.

Exp. 111. Draw the net and make a model of a rectangular block of length 2·6 in., breadth 1·9 in., and depth 1·4 in.

Exp. 112. Draw the net and make a model of a rectangular block whose length, breadth and depth are all equal to one another—say 2 in. Call it a cube.

Exp. 113. In any rectangular block state :—

(a) The number of faces.
(b) The number of edges.
(c) The number of corners.
(d) The shape of each face.
(e) The number of edges that meet at each corner.
(f) The number of faces that meet at each corner.

Figs. 34, 35, 36 represent a solid which we shall call a **pyramid.**

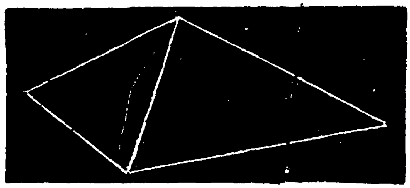


FIG. 34.

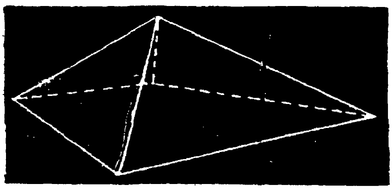


FIG. 35.

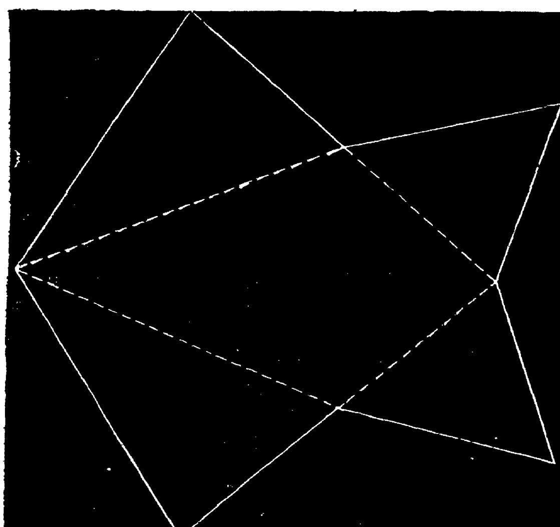


FIG. 36.

A pyramid is triangular, quadrilateral, pentagonal, hexagonal, etc., according as its base (that is, the face upon which it stands) is bounded by 3, 4, 5, 6, etc., straight lines respectively.

Exp. 114. Fig. 36 is the net of a pyramid. Copy it on thin cardboard and make a model of the pyramid by cutting out the net, bending along the dotted lines, and joining the edges with gummed paper.

Exp. 115. Fig. 39 is the net of a pyramid standing on a regular polygon (that is, a polygon whose sides and angles are equal) as base and having all its side edges equal to one another. Copy it and make a model of the solid and call it a right regular pyramid.

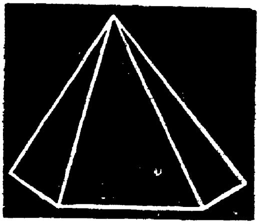


Fig. 37.

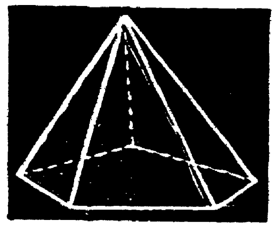


Fig. 38.

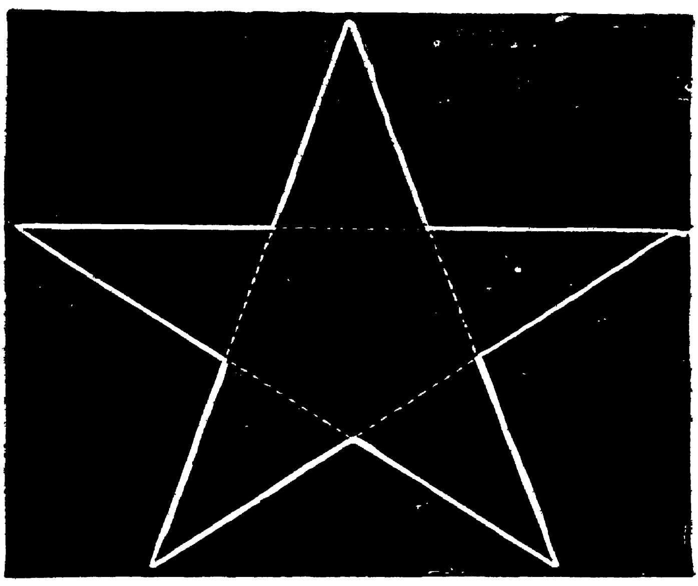


Fig 39

Exp. 116. Fig. 42 is the net of a pyramid standing on a triangular

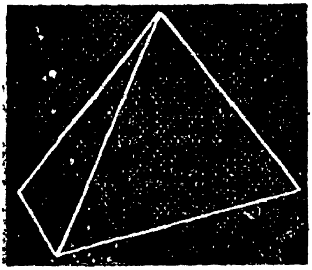


FIG. 40.

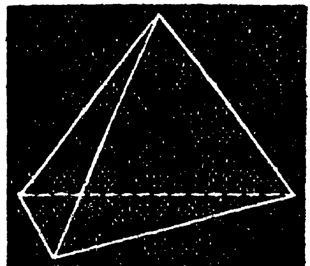


FIG. 41.

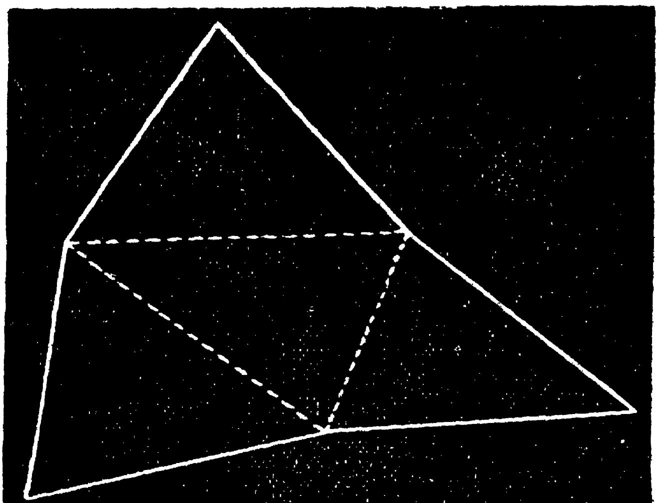


FIG. 42.

base. Copy it and make a model of the solid and call it a tetrahedron (or triangular pyramid).

Exp. 117. Draw the net and make a model of a tetrahedron having all its edges equal to one another. Call it a regular tetrahedron.

Exp. 118. In any tetrahedron state :—

(a) The number of faces.
(b) The number of edges.
(c) The number of corners.
(d) The shape of each face.
(e) The number of edges that meet at each corner.
(f) The number of faces that meet at each corner.

FIG. 43.

FIG. 44.

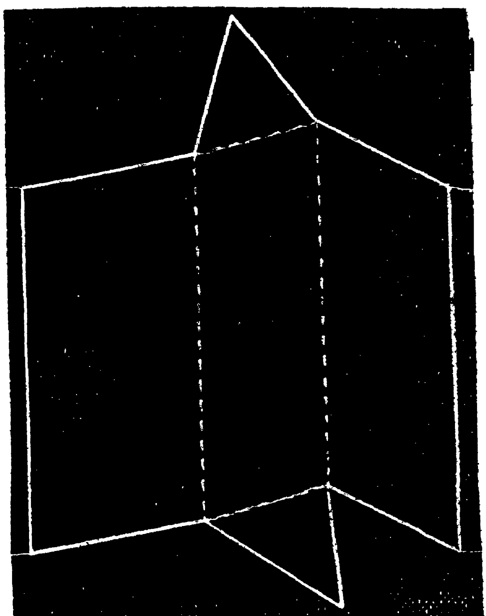


FIG. 45.

Figs 43, 44, 45 represent a solid which we shall call a prism.

A prism is triangular, quadrilateral, pentagonal, hexagonal, etc., according as its base is bounded by three, four, five, six, etc., straight lines respectively.

Exp. 119. Fig. 45 is the net of a prism. Copy it and make a model of the solid.

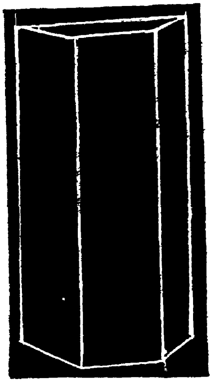


FIG. 46.

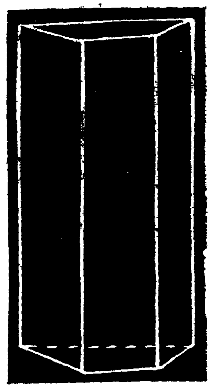


FIG. 47.

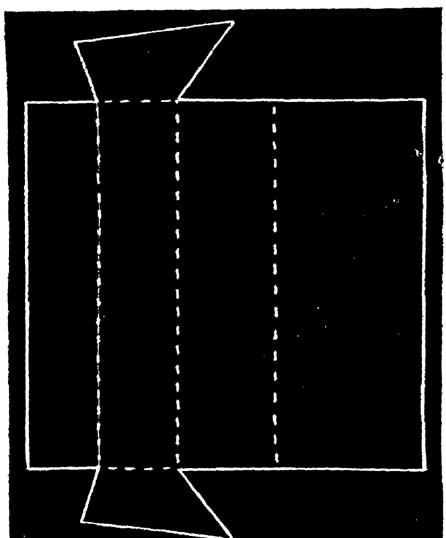


FIG 48.

Exp. 120. Fig. 48 is the net of a prism having all its side edges upright to the base. Copy it and make a model of the solid and call it a right prism.

Exp. 121. Draw the net and make a model of a right triangular prism or wedge.

Exp. 122. In any right triangular prism state :—

(*a*) The number of faces.
(*b*) The number of edges.
(*c*) The number of corners.
(*d*) The shape of the end faces.
(*e*) The shape of the side faces.
(*f*) The number of edges that meet at each corner.
(*g*) The number of faces that meet at each corner.

It is important to remember that so far we have not *proved* any Geometrical Truths, we have only *illustrated* them.

THEORETICAL SECTION.

INTRODUCTORY.

THINK of some solid body—a common brick for instance. It has length, breadth and thickness, and these we call its three dimensions. Now imagine one of these three dimensions—its thickness—to be gradually cut away until it disappears altogether. There will then be nothing left but a face of the brick, and this is an example of what we understand by a geometrical surface. Hence a surface has only two dimensions—length and breadth—and it may be regarded as a boundary of a portion of space. Now imagine this face of the brick to have its breadth gradually cut away. When all its breadth has gone there will be nothing left but an edge of the brick, and this is an example of what we understand by a geometrical line. Hence a line has only one dimension—length—and it may be regarded as a boundary of a surface. Finally, imagine this edge of the brick to have its length gradually cut away. When it has lost all its length there will be nothing left but a corner of the brick, and this is an example of what we understand by a geometrical point. Hence a point has no dimensions, and it may be regarded as a boundary or extremity of a line.

We will now define some terms of frequent use in Geometry.

DEFINITIONS.

Def. 1. A point is that which has position, but no length, breadth or thickness.

Def. 2. A line has position and length, but no breadth or thickness.

The extremities of a line are points.

If the extremities of a line are fixed, the line is said to be finite or limited; if they are not fixed, it is said to be infinite or unlimited.

The intersection of two lines is a point or points. This suggests a convenient way of indicating a point on paper. For example,

× P

is a better way of indicating the point P than

• P.

The path (or trace) of a moving point is a line.

The extent of a line is called its length.

Def. 3. A straight line lies evenly between its extreme points.

This definition is not altogether satisfactory, for it only amounts to substituting the word "evenly" for the word "straight," and nothing is gained in clearness. As a matter of fact the idea of a straight line is familiar to everyone, and cannot be made more so by definition. But straight lines have certain properties which it is well to bear in mind. These are they:—

(i) *Two straight lines cannot enclose a space.*

(ii) *Any part of a straight line will, however placed, lie wholly on any other part if its extremities are made to fall on that other part.*

(iii) *A straight line is the shortest distance between its extreme points.*

If a line is not straight it is said to be curved.

Def. 4. A surface has position, length and breadth, but no thickness.

The boundaries of surfaces are lines.
The intersection of two surfaces is a line or lines.
The path (or trace) of a moving line is generally a surface.

The extent of a surface is called its area.

Def. 5. A plane surface or plane is a surface in which any two points being taken the straight line that joins them lies wholly in that surface.

The intersection of two planes is a straight line.

If a surface is not plane it is said to be curved.

Def. 6. A solid has position, length, breadth and thickness.

The boundaries of solids are surfaces.
The path (or trace) of a moving surface is generally a solid.
Notice that the material of a solid does not enter into this definition, so that a bubble or a cloud is quite as much a solid geometrically as a lump of lead. Hence a solid may be defined as a limited portion of space.

The extent of a solid is called its volume.

QUESTIONS FOR EXAMINATION.—I.

Definitions 1 to 6.

1. Why is a star not a point? (Ans. Because it has length, breadth and thickness.)

2. Why is a piece of string not a line? (Ans. Because it has breadth and thickness.)

3. Is a shadow a surface? (Ans. Yes, because it has position, length and breadth, but no thickness.)

4. Say whether each of the following is a point, a line, a surface, or a solid:—

(a) A sheet of paper. (Ans. A solid.)
(b) The centre of the earth. (Ans. A point.)

(*c*) A coat of paint. (Ans. A solid.)
(*d*) The equator. (Ans. A line.)
(*e*) The North Pole. (Ans. A point.)
(*f*) The boundary between the air and water in a tumbler. (Ans. A surface.)
(*g*) The crease in a piece of paper that has been folded and opened out again. (Ans. A line, because it is the intersection of two surfaces.)

5. If a straight edge can be made to lie along a surface in every direction, why do we know that the surface is a plane? (Ans. Because any two points being taken in the surface the straight line that joins them lies wholly in the surface.)

6. By how many surfaces is a common brick bounded? (Ans. Six.)

7. Name a solid that is bounded by:—
(*a*) Four surfaces. (Ans. A tetrahedron.)
(*b*) Three surfaces. (Ans. A round ruler.)
(*c*) Two surfaces. (Ans. Half an orange.)
(*d*) One surface. (Ans. A cricket ball.)

DEFINITIONS.

Def. 7. When two straight lines meet in a point they are said to make with each other a plane rectilineal angle or, shortly, an angle.

The idea of an angle is fundamental and so cannot be satisfactorily defined. But we can illustrate the nature of an angle as follows:—

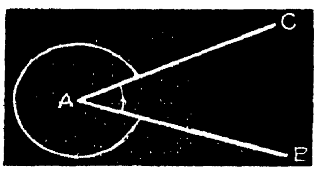


FIG. 49.

Let **AB** and **AC** meet at **A**. A line may turn about **A** from the position occupied by **AB** until it reaches the position occupied by **AC**. This revolving line is said to turn through the angle made by **AB** with **AC**, and *the size of the angle depends entirely upon the amount of turning* and not in any way upon the lengths of **AB**, **AC** or the revolving line.

Notice that the revolving line can turn from one position to the other in either of the two different ways indicated by the arrows. Hence two different angles are made by **AB** with **AC**, but the smaller angle is the one usually understood as the angle made by **AB** with **AC**. The greater angle is said to be **reflex** or **re-entrant**.

The angle made by **AB** with **AC** is denoted by the letters **BAC** or **CAB** or simply **A** or by inserting a Greek letter.

The point at which two straight lines meet to form an angle is called the **vertex** of the angle or the **angular point**, and the lines themselves are called the **arms** of the angle.

When the arms of an angle are in the same straight line but in

FIG. 50.

opposite directions, the angle is sometimes called a **straight angle**.

Def. 8. Two angles are said to be **adjacent** when they have a com-

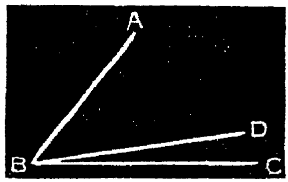


FIG. 51.

mon vertex and lie on opposite sides of a common arm. Thus the angles ABD, DBC are adjacent.

Since the angles ABD, DBC together make up the angle ABC, the angle ABC is the sum of the angles ABD and DBC, and, therefore, the angle DBC is the difference of the angles ABC and ABD.

Def. 9. A **plane figure** is a part of a plane bounded by one or more lines.

When the boundaries of a plane figure are all straight lines they are said to enclose a **plane rectilineal figure** and are called its **sides**.
A plane rectilineal figure is said to be:—

> **Equilateral** when all its sides are equal to one another.
> **Equiangular** when all its angles are equal to one another.
> **Regular** when it is both equilateral and equiangular.

The sum of the boundaries of a plane figure is called its **perimeter**.
The term "figure" is also used to denote points lines, surfaces and any combination of these.

We shall for the present confine ourselves to the study of figures in a plane—a branch of Geometry called **Plane Geometry**.

QUESTIONS FOR EXAMINATION.—II.

DEFINITIONS 1 TO 9.

1. Why do we know that only one straight line can be drawn between two given points? (Ans. Because two straight lines cannot enclose a space.)

2. Which is the most convenient way of showing the straight line from one corner of a room to another? (Ans. By stretching a string, because a straight line is the shortest distance between its extreme points.)

3. Write down the angle at B in ten different ways.

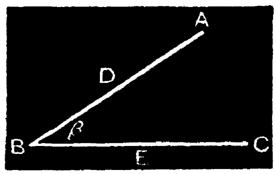


FIG. 52.

(Ans. ABC, CBA, ABE, EBA, DBC, CBD, DBE, EBD, B and β.)

4. In the accompanying figure name the four pairs of adjacent angles.

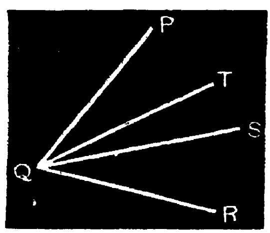


FIG. 53.

(Ans. PQT and TQS, TQS and SQR, PQS and SQR, PQT and TQR.)

5. Show that the size of an angle does not depend on the lengths of its arms. (See Def. 7.)

6. In what sense is an angle not a figure? (Ans. It is not bounded or enclosed by its arms.)

7. How many angles are there inside a ten-sided rectilineal figure? (Ans. Ten.)

Definitions.

Def. 10. When a straight line standing on another straight line makes the adjacent angles equal to one another each of these angles

Fig. 54.

is called a right angle and the straight line which stands on the other is called a perpendicular to it.

If a line, rotating about an extremity, makes a quarter of a complete revolution, it turns through a right angle.

The ninetieth part of a right angle is called a degree (written 1°). The sixtieth part of a degree is called a minute (written 1′). The sixtieth part of a minute is called a second (written 1″).

Def. 11. An obtuse angle is greater than one right angle but less than two right angles.

Fig. 55.

Def. 12. An acute angle is less than a right angle.

Fig. 56.

Def. 13. A circle is a plane figure contained by one line which is called the circumference, and is such that all straight lines drawn

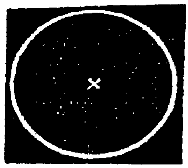


FIG. 57.

from a certain point within the figure to the circumference are equal to one another. This point is called the centre of the circle.

The word "circle" is used in geometry sometimes to mean the whole figure and sometimes only its circumference.

Def. 14. Any straight line drawn from the centre of a circle to its circumference is called a radius of the circle, and any straight line

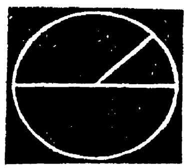


FIG. 58.

drawn through the centre of a circle and terminated both ways by the circumference is called a diameter of the circle.

Hence a diameter is double a radius of the same circle.

Two circles are equal if the radii of one are equal to the radii of the other.

Def. 15. The bisector of a magnitude is that which divides it into two equal parts—the trisectors into three equal parts.

It is obvious that a finite straight line has only one point of bisection, and only one straight line can be drawn to bisect a given plane rectilineal angle.

QUESTIONS FOR EXAMINATION.—III.

DEFINITIONS 1 TO 15.

1. Can you think of a smooth surface which is not a plane? (Ans. Yes—the surface of a perfectly polished billiard ball.)

2. How many straight lines at least are needed to bound a figure? (Ans. Three.)

3. How many times are the hands of a clock at right angles in the course of an hour? (Ans. Twice.)

4. Prove that any point on a hand of a clock describes a circle. (Ans. Because its path is always at the same distance from the centre of the clock-face.)

5. Name the angle that is left in the accompanying figure if we take

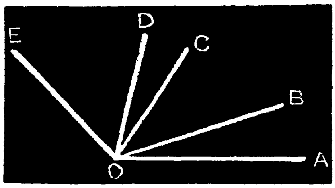


FIG. 59.

away the sum of the angles **BOC** and **COD** from the difference of the angles **AOE** and **DOE**. (Ans. The angle **AOB**.)

6. Name a figure having only one boundary. (Ans. A circle.)

CONCERNING THE POSTULATES.

Geometrical truths are proved by means of figures. To draw these figures it must be granted that certain simple constructions are possible, but the fewer they are the better. And they must be so simple that their possibility is self-evident and requires no proof. We shall make three of these demands and call them our **postulates**. These are they :—

THE POSTULATES.

Let it be granted :—

1. *That a straight line may be drawn from any one point to any other point.*
2. *That a finite straight line may be produced to any length in that straight line.*
3. *That a circle may be described with any centre and with a radius equal to any finite straight line.*

These postulates practically ask that we may use a ruler or straight-edge and compasses, and that the straight lines and circles drawn by means of these instruments may be regarded as satisfying our definitions. As a matter of fact, we know how imperfect they are however skilfully they may be drawn, but that is no reason why they should not be *assumed* to be ideally correct.

Notice that the postulates do not ask to use the ruler for comparing the lengths of lines, but the compasses may be used for carrying lengths from place to place. We may assume that the

following simple constructions are possible because they follow immediately from postulates 2 and 3 :—

(*a*) *From a given point to draw a straight line equal to a given straight line.*

(*b*) *From the greater of two given straight lines to cut off a part equal to the less.*

QUESTIONS FOR EXAMINATION.—IV.

DEFINITIONS 1 TO 15 AND THE POSTULATES.

1. If two straight lines have their extremities in common, they must coincide. Why do we know this? (Ans. Because two straight lines cannot enclose a space.)

2. Prove that two right angles are together greater than twice any acute angle. (Ans. Because an acute angle is less than a right angle.)

3. Prove that any two obtuse angles are together greater than twice a right angle. (Ans. Because an obtuse angle is greater than a right angle.)

4. How would you test geometrically whether a surface is plane or not? (Ans. By taking two points in it and seeing if the straight line joining them lies wholly in the surface wherever the points be taken.)

5. How would you test geometrically whether a line is straight or not? (Ans. If it is straight it cannot be made to enclose a space with another line that we know to be straight.)

6. Why are a certain number of postulates necessary? (Ans. To enable us to draw the figures that are necessary for the proofs of geometrical truths.)

7. What demands do the postulates make beyond the use of ruler and compasses? (Ans. That the figures drawn by these instruments be considered geometrically correct.)

CONCERNING THE AXIOMS.

To prove a statement we must show that it depends upon and can be inferred from some other statement or statements that have been already proved or that are self-evident. It is clear, then, that before we begin to reason about figures we must assume the truth of certain simple statements. They must be as few as possible and so simple that they do not depend for their truth upon statements that are still simpler. In other words, it must be impossible to prove them. These self-evident truths are called **axioms**. They form the basis of the whole science of geometry, and all geometrical truths are derived from them by a process of reasoning called **deduction**.

We shall arrange our axioms under two headings :—

(*a*) **General axioms**, which are true of magnitudes of all kinds.

(*b*) **Geometrical axioms**, which are true of geometrical magnitudes only.

General Axioms.

Ax. 1. *Things which are equal to the same thing are equal to one another.*

Ax. 2. *If equals be added to equals the wholes are equal.*

As a particular case of this axiom we have :—

Things which are doubles of the same thing are equal to one another.

Ax. 3. *If equals be taken from equals the remainders are equal.*

Ax. 4. *If equals be added to unequals the wholes are unequal.*

Ax. 5. *If equals be taken from unequals the remainders are unequal.*

Ax. 6. *Things which are halves of the same thing are equal to one another.*

Ax. 7. *The whole is equal to the sum of its parts.*

Hence the whole is greater than its part.

Geometrical Axioms.

Ax. 8. *Any magnitude can be transferred from one position to another without its shape or size undergoing any change.*

Ax. 9. *Magnitudes which coincide with one another, that is which exactly fill up the same space, are equal to one another.*

Notice that it is not sufficient for the magnitudes to fill up equal spaces. They must fill up *the very same space.*

The process of applying one magnitude to another for the purpose of comparison is called superposition.

Ax. 10. *Two straight lines cannot enclose a space.*

Ax. 11. *All right angles are equal.*

Ax. 12. *See page* 56.

QUESTIONS FOR EXAMINATION.—V.

Definitions 1 to 15, the Postulates and the Axioms.

1. How many dimensions of space have the following ?—

   (*a*) A circle. (Ans. Two.)
   (*b*) The centre of a circle. (Ans. None.)
   (*c*) The circumference of a circle. (Ans. One.)
   (*d*) A tennis ball. (Ans. Three.)

2. Within what limits do the postulates demand the use of ruler and compasses ? (Ans. That they be used for joining points, producing lines and describing circles, but not for measuring lengths.)

3. Why are a certain number of axioms necessary ? (Ans. To give grounds for deducing other geometrical truths that are not so self-evident.)

4. Which axioms do the following reasonings illustrate?—
   (*a*) Two men are the same age. If they both live they will be the same age twenty years hence. (Ans. Ax. 2.)
   (*b*) Three bullets are cast in the same mould, hence all three bullets will be equal to one another. (Ans. Ax. 9.)
   (*c*) A is richer than B. If each loses £50 one will still be richer than the other. (Ans. Ax. 5.)

5. Prove that the radius of a circle is less than the straight line joining the centre of the circle to any point outside the circle. (Ans. Because a radius can be drawn which, when produced, will pass through the outside point, and the whole is greater than its part.)

6. In what two ways can the equality of geometrical magnitudes be tested? (Ans. (i) By seeing if they are equal to the same thing (Ax. 1); (ii) By seeing if they can be made to coincide with one another (Ax. 9).)

7. An angle **ABC** is folded so that **BC** lies along **BA** and then opened out. Why do we know that the crease divides the angle into two equal parts? (Ans. Because one part can be exactly superposed over the other.)

8. **C** is a point in the straight line **AB**. The paper is folded so that **CB** lies along **CA** and then opened out. Why do we know that the crease is perpendicular to **AB**? (Ans. Because the crease makes angles with **AB** which we know to be equal since one can be exactly superposed over the other.)

THE PROPOSITIONS.

We are now prepared to enter upon a series of separate discussions called propositions, relating to the terms that we have defined and regulated by the postulates and axioms that we have laid down. These propositions are of two kinds: theorems and problems.

A theorem is a proposition in which some geometrical property is required to be *demonstrated*, *e.g.*:—

If two straight lines intersect, the vertically opposite angles are equal.

A problem is a proposition in which some geometrical construction is required to be *effected*, *e.g.*:—

Bisect a given finite straight line.

Remember, then, that in a theorem something has to be *proved*, while in a problem something has to be *done*. The statement or enunciation of a proposition is naturally divided into two parts. In the case of a theorem it consists of the **hypothesis** or conditions assumed, and the conclusion or assertion to be proved. Thus in the theorem, "If two straight lines intersect, the vertically opposite angles are equal," we have:—

Hypothesis—It is assumed that two straight lines intersect.

Conclusion—It is to be proved that the vertically opposite angles are equal.

In the case of a problem the enunciation consists of the data or things given and the quæsita or things required. Thus in the problem, "Bisect a given finite straight line," we have:—

Datum—There is given a finite straight line.

Quæsitum—It is required to bisect it.

Every proposition consists of four parts:—

I. The general enunciation, which states in general terms the conditions of the theorem or problem.

II. The particular enunciation, which repeats the general enunciation in special terms and makes it refer to some particular figure.

III. The construction, in which such lines are drawn, on the authority of the postulates, as may be necessary to prepare the figure for demonstration.

IV. The demonstration or proof, which shows that the theorem is true or the problem is possible, as illustrated by the figure.

Problems belong to Practical Geometry, and will be considered under that branch of the subject.

SYMBOLS.

$+$ for plus.
$-$ „ minus.
$\angle$ „ angle.
$=$ „ is equal to, are equal to, equal to, equal.
$\perp$ „ perpendicular.
$\parallel$ „ parallel.
$\triangle$ „ triangle.
$\equiv$ „ is equal in all respects to, equal in all respects to.

$>$ for is greater than, are greater than, greater than.
$<$ „ is less than, are less than, less than.
$\odot$ „ circle.
$\because$ „ because.
$\therefore$ „ therefore.
$\square^{gm}$ „ parallelogram.

A symbol followed by the letter *s* denotes that it is being used in the plural.

ABBREVIATIONS.

st. for straight.
rt. „ right.
sq. „ square.
pt. „ point.
join AB (for example) for draw a straight line from A to B.
hyp. for hypothesis.

hypot. for hypotenuse.
Q.E.D. „ quod erat demonstrandum (which was to be proved).
Q.E.F. „ quod erat faciendum (which was to be done).

And all obvious contractions such as *quadl.* for "quadrilateral" and *equiang.* for "equiangular".

THEOREMS.

In proving the following theorems we shall assume that certain constructions are possible, though they fall outside the scope of the postulates.

These hypothetical constructions are :—

(*a*) *A line or angle can be divided into any number of equal parts.*

(*b*) *A line can be drawn from any point in any desired direction and of any desired length.*

(*c*) *A figure can be reproduced or placed in any position.*

Though we may not know *how* to perform these constructions, we can perfectly easily imagine them as *having been* performed.

ANGLES AT A POINT.

Theorem 1.

(Euc. I. 13.)

Gen. Enun. *If a straight line stands on another straight line the sum of the two angles so formed is equal to two right angles.*

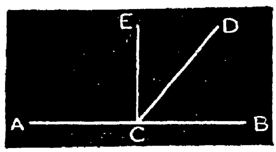


Fig. 60.

Part. Enun. Let the st. line DC stand on the st. line AB so as to form the 2 $\angle$s BCD and DCA.

It is reqd. to prove that

$$\angle BCD + \angle DCA = 2 \text{ rt. } \angle s.$$

Const. Suppose CE to have been drawn $\perp$ to AB.

Proof. $\angle BCD + \angle DCA = \angle BCD + \angle DCE + \angle ECA.$

$= \angle BCE + \angle ECA.$

$= 2$ rt. $\angle$s. Q.E.D.

QUESTIONS FOR EXAMINATION.—VI.

1. Distinguish between a theorem and a problem.
2. Name the four parts of a proposition and state the purpose of each part.
3. What is the hypothesis in Theorem 1?
4. What is the conclusion in Theorem 1?
5. Repeat Axiom 7. What use is made of Axiom 7 in Theorem 1?
6. What hypothetical construction is employed in Theorem 1?
7. Prove Theorem 1 by rotating a line about C from the position CB to the position CD and thence to the position CA, remembering that a right angle is a quarter of a complete revolution.

DEFINITIONS.

Def. 16. When two angles are together equal to two right angles, each is said to be the supplement of the other and the two angles are said to be supplementary.

Thus in Fig. 60, the angle ACD is the supplement of the angle DCB.

Def. 17. When two angles are together equal to one right angle, each is said to be the complement of the other and the two angles are said to be complementary.

Thus in Fig. 60, the angle BCD is the complement of the angle DCE.

Def. 18. A corollary is a geometrical truth that can be easily deduced from a proved proposition.

We shall now give some corollaries on Theorem 1 and some exercises to be worked out by the student.

Cor. 1. *If two straight lines cut one another, the four angles they make at the point where they cut are together equal to four right angles.*

Cor. 2. *All the angles made by any number of straight lines that meet at a point, taken in order, are together equal to four right angles.*

Exercises.

1. What is the complement of each of the following ∟s?—
   (i) 17°. (Ans. 73°.)
   (ii) 29° 13′. (Ans. 60° 47′.)
   (iii) 82° 26′ 15″. (Ans. 7° 33′ 45″.)
2. What is the supplement of each of the following ∟s?—
   (i) 73°. (Ans. 107°.)
   (ii) 35° 57′. (Ans. 144° 3′.)
   (iii) 115° 19′ 42″. (Ans. 64° 40′ 18″.)
3. If several st. lines stand upon another st. line at the same pt. the consecutive ∟s so formed are together = 2 rt. ∟s. (See Fig. 61.)

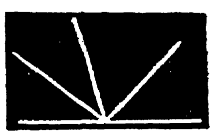


FIG. 61.

4. The supplement of an acute ∟ is obtuse.
5. ∟s that are complements of the same ∟ are = to one another.
6. ∟s that are supplements of the same ∟ are = to one another.
7. If there are 15 rays in a perfect star, show that the angle between 2 consecutive rays is 24°.

8. There are 12 spokes in a wheel. Find the $\angle$ between 2 consecutive spokes. (30°.)

9. Of the 4 $\angle$s formed by 2 st. lines cutting one another, if one is a rt. $\angle$ they are all rt. $\angle$s.

10. Can an $\angle$ = its own supplement, and, if so, how many degs. are there in it? (Yes. 90°.)

11. Can an $\angle$ = its own complement, and, if so, how many degs. are there in it? (Yes. 45°.)

12. Can an $\angle$ be double its own supplement, and, if so, how many degs. are there in it? (Yes. 120°.)

13. Can an $\angle$ be $\frac{1}{5}$ of its own complement, and, if so, how many degs. are there in it? (Yes. 15°.)

14. *If a st. line stands on another st. line the bisectors of the 2 $\angle$s so formed are $\perp$ to one another.* (Bom. Schl. Final.)

15. If $\angle$ AOB = 72°, how many degs. are there in the reflex $\angle$ made by OA with OB? (288°.)

16. BD is the bisector of $\angle$ ABC and DB is produced to E. Prove that $\angle$ ABE = $\angle$ CBE.

17. The internal and external bisectors of an angle make with one another an $\angle$ = $\frac{1}{2}$ the sum of the $\angle$ and its supplement. NOTE.—If the arm CB of $\angle$ ABC is produced to D, the bisector of $\angle$ ABD is called the external bisector of $\angle$ ABC.

*18. In Fig. 60 prove that $\angle$ ECD is $\frac{1}{2}$ of the difference of $\angle$s BCD, DCA.

Theorem 2.

(Euc. I. 14.)

Gen. Enun. *If at a point in a straight line two other straight lines on opposite sides of it make the adjacent angles together equal to two right angles these two straight lines are in one and the same straight line.*

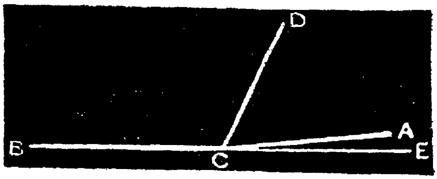


Fig. 62.

Part. Enun. At the point C in the st. line CD and on opposite sides of it, let the st. lines CA, CB make the adj. ∟s DCA, DCB together = 2 rt. ∟s.

It is reqd. to prove that

CA *and* CB *are in one and the same st. line.*

Const. Produce BC to E.

Proof. ∵ DC stands upon the st. line ECB,

∴ ∟DCE + ∟DCB = 2 rt. ∟s . . . Th. 1.

But ∟DCA + ∟DCB = 2 rt. ∟s . . . Hyp.

∴ ∟DCE + ∟DCB = ∟DCA + ∟DCB.

∴ ∟DCE = ∟DCA.

∴ CA coincides with CE.

But, by construction, CE and CB are in one and the same st. line.

∴ CA and CB are in one and the same st. line. Q.E.D.

Def. 19. Two theorems are said to be converse each of the other when the hypothesis of each is the conclusion of the other.

This is true in the case of Theorems 1 and 2, for in Theorem 1 we assume that two straight lines meet one another and we prove that two adjacent angles formed by them are together equal to two right angles, while in Theorem 2 we assume that two adjacent angles are together equal to two right angles and we prove that they are formed by two straight lines meeting one another.

The converse of a theorem need not be true though the theorem itself may be true. For example, the enunciation,

"If a man is a negro, he must have a dark skin,"

is true, but its converse,

"If a man has a dark skin, he must be a negro,"

is evidently not true.

The converse of a theorem whose hypothesis is made up of several distinct hypotheses and whose conclusion is made up of several distinct conclusions is obtained by interchanging one of the conclusions and one of the hypotheses. For further explanation see note on Theorem 14.

QUESTIONS FOR EXAMINATION.—VII.

1. Into what two parts can the enunciation of a theorem be divided?

2. If the words "on opposite sides of it" were omitted, would the enunciation of Theorem 2 still hold good?

3. Repeat Axiom 1 and show what use is made of it in the proof of Theorem 2. What other axiom is used?

4. When are two theorems said to be each the converse of the other?

5. State the converse of the enunciation:—

"All birds have wings".

6. In the proof of Theorem 2 we assume that if two angles are equal their arms can be made to coincide. Prove this by superposition.

Exercises.

19. OA, OB, OC, OD meet at O. The $\angle$s AOB, BOC, COD, DOA are all rt. $\angle$s. Prove that OA is in the same st. line as OC and OB as OD.

20. OA, OB, OC, OD meet at O. $\angle$ AOB + $\angle$ BOC = $\angle$ COD + $\angle$ DOA. Prove that OC is in the same st. line as OA.

21. OA, OB, OC, OD meet at O. $\angle$ AOB = $\angle$ COD and $\angle$ BOC = $\angle$ DOA. Prove that OC is in the same st. line as OA and OB as OD.

22. If the bisectors of 2 adj. $\angle$s are $\perp$ to one another, prove that the adj. $\angle$s are formed by 2 st. lines meeting one another.

23. At the pt. C in the st. line AB, 2 st. lines CD, CE on opp. sides of it make $\angle$ DCA = $\angle$ ECB. Prove that CD and CE are in the same st. line. (Punj. Eur. Schls. Mid.)

*24. AOB is an acute $\angle$. Prove that the bisectors of the acute $\angle$ AOB and of the reflex $\angle$ AOB are in one and the same st. line.

Def. 20. Of the four angles formed by two intersecting st. lines those that are opposite one another are called vertically opposite angles.

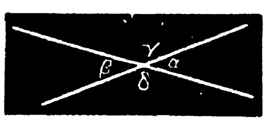


FIG. 63.

Thus, in Fig. 63, α and β are vertically opposite angles and so are γ and δ.

Theorem 3.

(Euc. I. 15.)

Gen. Enun. ***If two straight lines intersect, the vertically opposite angles are equal.***

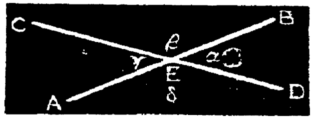


Fig. 64.

Part. Enun. Let the st. lines **AB**, **CD** intersect at **E**, forming the $\angle$s $\alpha, \beta, \gamma, \delta$.

It is reqd. to prove that

$$\angle\alpha = \textit{vert. opp.} \angle\gamma.$$
$$\text{and } \angle\beta = \textit{vert. opp.} \angle\delta$$

Proof. $\because$ **BE** stands on **CD**,

$\therefore \angle\alpha + \angle\beta = 2$ rt. $\angle$s. . . Th. 1.

Again $\because$ **CE** stands on **AB**,

$\therefore \angle\beta + \angle\gamma = 2$ rt. $\angle$s. . . Th. 1.

$\therefore \angle\alpha + \angle\beta = \angle\beta + \angle\gamma,$

$\therefore \angle\alpha = \angle\gamma,$

Similarly $\angle\beta = \angle\delta.$ Q.E.D

QUESTIONS FOR EXAMINATION.—VIII.

1. Why do we know that two straight lines cannot intersect in more than one point? (Ans. Because two straight lines cannot enclose a space.)
2. Define "vertically opposite" angles.
3. Repeat Axiom 11, and show what use is made of Axiom 11 in the proof of Theorem 3.
4. Give the hypothesis and the conclusion in the enrnciation, "All men will die".
5. If two straight lines meet at a point, how many angles do they make with one another? (Ans. 2.)
6. From the defn. of a line prove that the intersection of two lines is a point or points.

Exercises.

25. How many degs. are there in the supplement of $\frac{3}{5}$ of a rt. $\angle$ (Ans. 126°.)

26. In Fig. 64 prove that $\angle\beta = \angle\delta$.

27. *OA and OB meet at O. OC and OD are drawn from O $\perp$ to OA and OB respectively. Prove that $\angle$ COD is either equal or supplementary to $\angle$ AOB.*

28. Two adj. $\angle$s are supplementary. How many degs. are there in the $\angle$ formed by their bisectors? (Ans. 90°.)

29. *The bisectors of vert. opp. $\angle$s are in one and the same st. line.*

*30. **AB** and **CD** intersect at **E**. **EF** is the bisector of $\angle$ **AEC**. Prove that **FE** produced will bisect the vert. opp. $\angle$ **DEB**.

PARALLEL STRAIGHT LINES.

Def. 21. Straight lines, in the same plane, which do not meet however far they are produced in either direction, are said to be parallel.

Note.—If a straight line cuts two other straight lines, eight angles are formed at the two points of intersection, as in Fig. 65. Of these

α, β, θ, η are called exterior angles.
γ, δ, ϵ, ζ are called interior angles
γ, ϵ and δ, ζ are called alternate angles.
α, ϵ; β, ζ; η, δ and θ, γ are called corresponding angles.

The cutting line is called a transversal.

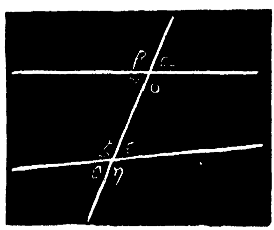


Fig. 65.

THEOREM 4.*

(Euc. I. 27.)

Gen. Enun. *When a straight line cuts two other straight lines, if a pair of alternate angles are equal, then the two straight lines are parallel.*

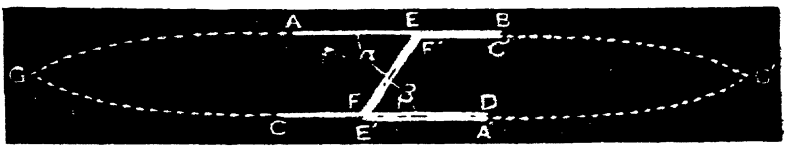


FIG. 66.

Part. Enun. Let the st. line EF cut the two other st. lines AB, CD in E, F respy. and make ∟ α = alt. ∟ β.

It is reqd. to prove that

AB *is* ∥ *to* CD.

Const. If AB and CD are not ∥, they will meet when produced either in one direction or in the other.

Suppose they meet at G when produced in the direction of A and C.

Now imagine a copy, G′A′E′F′C′ to be made of the fig. GAEFC on tracing paper, and this copy to be turned round in its own plane and to take up a position so that E′F′ may coincide with FE.

Proof. ∵ ∟ A′E′F′ = ∟ AEF = ∟ α

= ∟ β, . . . Hyp.

∴ E′A′ will lie along FD.

And ∵ ∟ E′F′C′ = ∟ EFC

= supplement of ∟ β . Th. 1.

= supplement of ∟ α . Hyp.

= ∟ FEB . . . Th. 1.

∴ F′C′ will lie along EB.

Hence EB and FD when produced in the direction of B and D will meet at G′.

But, if this were so, we should have two st. lines enclosing a space, which we know to be impossible.

∴ AB, CD cannot meet when produced in either direction.

∴ AB is ∥ to CD. Q.E.D.

* The Proof of this Theorem is difficult for a beginner, and may be omitted when reading the book for the first time.

ON REDUCTIO AD ABSURDUM.

It is sometimes easier to prove that a statement cannot be untrue than to prove that it is true. And these two conclusions come to the same thing. For example, if we prove that two given straight lines cannot be unequal, it is the same thing as proving that they are equal. A line of argument frequently used in geometry for proving the truth of a statement in this round-about manner is called the reductio ad absurdum (reducing to an absurdity). This method amounts to showing that *if* the statement whose truth we wish to prove is assumed to be untrue *then* impossible results must follow. It has been pointed out that the *reductio ad absurdum* only shows that the statement *is* true and not *why* it is true, and so is less valuable than a direct proof.

The *reductio ad absurdum* is used in the proof of Theorem 4, and is often useful in proving converse propositions.

QUESTIONS FOR EXAMINATION.—IX.

1. Repeat the definition of parallel straight lines and point out the necessity of the words "in the same plane".

2. Show by a diagram what you understand by "two alternate angles".

3. What name is given to the method of proof employed in Theorem 4? Why is it so called?

4. What false assumption led to the impossible conclusion in Theorem 4?

5. Repeat Axiom 8. What use is made of Axiom 8 in the proof of Theorem 4?

6. Enunciate the previous theorem that is employed in the proof of Theorem 4.

7. Prove by *reductio ad absurdum* that two straight lines cannot have a common segment.

Exercises.

31. If all the $\angle$s of a fig. bounded by 4 st. lines are rt. $\angle$s prove that its opp. sides are $\parallel$.

32. *If 2 st. lines are $\perp$ to the same st. line they are $\parallel$ to one another.*

33. The bisectors of either pair of alt. $\angle$s in Fig. 66 are $\parallel$.

34. When a st. line cuts 2 other st. lines if a pair of ext. but non-adj. $\angle$s, on opp. sides of the cutting line, are $=$, the 2 st. lines are $\parallel$.

*35. Prove that only one st. line can be drawn $\perp$ to a given st. line from a given pt. within it.

THEOREM 5.

(Euc. I. 28.)

Gen. Enun. *When a straight line cuts two other straight lines, if*

(A) *A pair of corresponding angles are equal, or*

(B) *A pair of interior angles on the same side of the cutting line are together equal to two right angles,*

then the two straight lines are parallel.

Part. Enun. Let the st. line EF cut the 2 other st. lines AB, CD and make (A) $\angle\alpha =$ corresp. $\angle\beta$.

or (B) int. $\angle$s β, γ on the same side of EF together = 2 rt. $\angle$s.

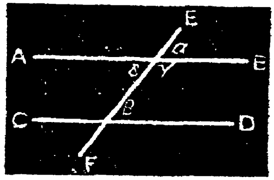


FIG. 67.

It is reqd. to prove that

AB *is* $\parallel$ *to* CD.

Proof of (A) $\angle\alpha =$ vert. opp. $\angle\delta$ . . . . Th. 3.

But $\angle\alpha =$ corresp. $\angle\beta$ . . . . Hyp.

$\therefore \angle\delta =$ alt. $\angle\beta$.

$\therefore$ AB is $\parallel$ to CD . . . . . Th. 4.

Q.E.D.

Proof of (B) $\angle\delta + \angle\gamma = 2$ rt. $\angle$s . . . . Th. 1.

Also $\angle\gamma + \angle\beta = 2$ rt. $\angle$s . . . . Hyp.

$\therefore \angle\delta + \angle\gamma = \angle\gamma + \angle\beta$.

$\therefore \angle\delta =$ alt. $\angle\beta$.

$\therefore$ AB is $\parallel$ to CD . . . . . Th. 4.

Q.E.D.

ON DIRECTION AND SENSE.

Straight lines that make equal angles with any line of reference as, for example, the four lines in Fig. 68, are said to have the same

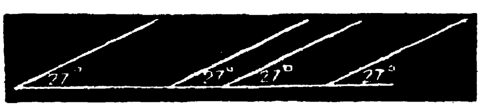


FIG. 68.

direction. But these angles that determine the directions of a set of straight lines are what we have called "corresponding angles".

Hence it follows from Theorem 5 that *lines having the same direction are parallel.*

Any line may be supposed to have been traced by a point moving from either extremity towards the other. We recognise this by saying that a line may be drawn in either of two opposite senses—for instance, from right to left or from left to right. So, too, a number of parallel straight lines can be drawn all in the same sense or some in one sense and some in another.

Again, a line rotating about an extremity may be supposed to move either clockwise or counter-clockwise. We recognise this by saying that a line rotating clockwise and a line rotating counter-clockwise are rotating in opposite senses.

Thus we can speak of two parallel straight lines or of two angles of rotation being "equal in magnitude and opposite in sense," or, briefly, "equal and opposite".

QUESTIONS FOR EXAMINATION.—X.

1. Repeat the definition of parallel straight lines and point out the necessity of the words "in either direction".
2. What axioms are employed in proving Theorem 5?
3. How can the direction of a straight line be determined?
4. Why do we know that AB and CD in Fig. 67 are in the same direction?
5. Prove that straight lines in the same direction are parallel to one another.
6. In Fig. 67 show that AB can be made to lie along CD by two equal rotations in opposite senses.
7. Prove by Theorem 5 that parallel straight lines can be correctly drawn by the aid of set-squares.

Exercises.

36. If all the ∠s of a fig. bounded by 4 st. lines are rt. ∠s, prove by Theorem 5 in 2 different ways that its opp. sides are ||.

37. When a st. line cuts 2 other st. lines, if a pair of *exterior* ∠s on the same side of the cutting line are together = 2 rt. ∠s, then the 2 st. lines are ||.

38. Prove Theorem 5 in the same way as Theorem 4.

Axiom 12 (Playfair's Axiom). *Two straight lines that intersect one another cannot both be parallel to the same straight line.*

For example, PQ and PR cannot both be parallel to AB because they intersect at P.

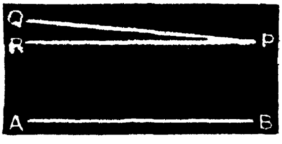


FIG. 69.

Theorem 6.

(Euc. I. 29.)

Gen. Enun. *If a straight line cuts two parallel straight lines,*
(A) *alternate angles are equal,*
(B) *corresponding angles are equal, and*
(C) *the interior angles on the same side of the cutting line are together equal to two right angles.*

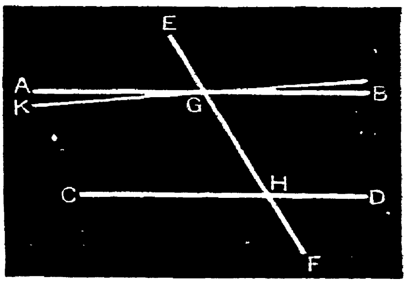


Fig. 70.

Part. Enun. Let the st. line **EF** cut the 2 || st. lines **AB, CD** in **G, H** respy.
It is reqd. to prove that
(A) $\angle$ **AGH** = *alt.* $\angle$ **GHD**,
(B) $\angle$ **EGB** = *corresp.* $\angle$ **GHD**, *and*
(C) *int.* $\angle$*s* **BGH, GHD** *on the same side of* **EF** *together* = 2 *rt.* $\angle$*s.*

Proof of (A) If $\angle$ **AGH** is not = $\angle$ **GHD**,
Suppose **GK** to have been drawn making $\angle$ **KGH** = alt. $\angle$ **GHD**.
$\therefore$ **KG** is || to **CD** . . . . . . Th. 4.
But **AG** is || to **CD** . . . . . . Hyp.
$\therefore$ **KG** and **AG** are both || to **CD** . . .
which is impossible . . . . Playfair's Axiom.
$\therefore$ $\angle$ **AGH** cannot be unequal to $\angle$ **GHD**,
that is, $\angle$ **AGH** = alt. $\angle$ **GHD**. Q.E.D.

Proof of (B) $\because$ $\angle$ **EGB** = vert. opp. $\angle$ **AGH** . . Th. 3.
and $\angle$ **AGH** = alt. $\angle$ **GHD** . . . Th. 6 (A).
$\therefore$ $\angle$ **EGB** = corresp. $\angle$ **GHD**. Q.E.D.

Proof of (C) $\because$ $\angle$ **BGH** + $\angle$ **AGH** = 2 rt. $\angle$s . . Th. 1.
and $\angle$ **AGH** = alt. $\angle$ **GHD** . . . Th. 6 (A).
$\therefore$ $\angle$ **BGH** + $\angle$ **GHD** = 2 rt. $\angle$s. Q.E.D.

Cor. *Two angles whose arms are parallel and drawn in the same sense are equal.*

QUESTIONS FOR EXAMINATION.—XI.

1. What false assumption is made in the proof of Theorem 6, and what impossible conclusion is the result?

2. Why is a *reductio ad absurdum* proof less valuable than a direct proof?

3. Prove that parallel straight lines have the same direction.

4. In Fig. 70, assuming that parallel straight lines have the same direction, prove that $\angle$ AGH = $\angle$ GHD.

5. Prove as a corollary of Theorem 6: "If a straight line cuts two other straight lines, so as to make the sum of the two interior angles on the same side of it less than two right angles, the two straight lines, if continually produced, will meet on that side on which are the angles whose sum is less than two right angles". (Euclid's Twelfth Axiom.)

6. What different kinds of magnitude are treated of in Plane Geometry?

Exercises.

39. A fig. is bounded by 4 st. lines of which opp. pairs are $\parallel$; prove that its 4 $\angle$s are together = 4 rt. $\angle$s.

40. If one of the 4 $\angle$s in the fig. of Ex. 39 is a rt. $\angle$, prove that all its $\angle$s are rt. $\angle$s. (Calc. F. A. Exam.)

41. Prove that the opp. $\angle$s in the fig. of Ex. 39 are = to one another.

42. *If* AB *is* $\perp$ *to* CD *it is also* $\perp$ *to all st. lines* $\parallel$ *to* CD.

43. AB, CD are $\perp$s to EF, GH respy. If EF is $\parallel$ to GH prove that AB is $\parallel$ to CD.

44. *The* $\angle$ *between 2 st. lines is = or supplementary to the* $\angle$ *between 2 others which are* $\perp$*s to them respy.*

45. If a st. line cuts 2 $\parallel$ st. lines, prove that the bisectors of alt. $\angle$s are $\parallel$.

*46. If a st. line cuts 2 $\parallel$ st. lines, prove that the bisectors of the 4 int. $\angle$s form a 4-sided fig. whose opp. sides are $\parallel$ and whose $\angle$s are all rt. $\angle$s.

*47. In the fig. of Ex. 39 prove that the bisectors of the $\angle$s form a 4-sided fig. whose opp. sides are $\parallel$ and whose $\angle$s are all rt. $\angle$s. (Bom. Prev.)

*48. Of 2 st. lines, if one is while the other is not $\perp$ to a third st. line, prove that these 2 st. lines cannot be $\parallel$.

*49. Prove, without producing a side, that the 3 $\angle$s of a $\triangle$ are together = 2 rt. $\angle$s. (Punj. F. E.)

THEOREM 7.

(Euc. I. 30.)

Gen. Enun. *Straight lines which are parallel to the same straight line are parallel to one another.*

Part. Enun. Let the st. lines AB, CD be each $\parallel$ to the st. line EF.

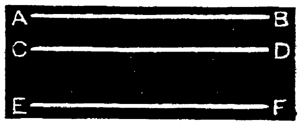


FIG. 71.

It is reqd. to prove that

AB *is* $\parallel$ *to* CD.

Proof. If AB and CD are not $\parallel$, they will intersect if produced far enough, and then two st. lines that intersect would both be $\parallel$ to the same st. line.

But this is impossible . . . . Playfair's Axiom.

$\therefore$ AB and CD will not intersect, however far produced.

$\therefore$ AB is $\parallel$ to CD. Q.E.D.

QUESTIONS FOR EXAMINATION.—XII.

1. State the converse of Theorem 7.

2. Repeat Playfair's Axiom. What use is made of it in the proof of Theorem 7?

3. If two straight lines are each parallel to a third straight line *that lies between them*, deduce immediately from the definition of parallel straight lines that they are parallel to one another.

4. What two kinds of surfaces are treated of in geometry?

5. What positive property has a geometrical point?

6. What do you understand by the expression $\angle$ AOB + $\angle$ BOC = $\angle$ AOC?

Exercises.

50. If a st. line is $\parallel$ to one of 2 $\parallel$ st. lines, it is $\parallel$ to the other.

51. If a st. line is not $\parallel$ to one of 2 $\parallel$ st. lines, it is not $\parallel$ to the other.

52. AB and AC are each $\parallel$ to DE. Prove that AB and AC are in one and the same st. line.

* 53. Prove Theorem 7 by means of Theorems 4 and 6 (Euclid's proof).

* 54. If 2 st. lines are not $\parallel$, prove that all st. lines cutting them make alt. $\angle$s which differ by the same $\angle$.

RECTILINEAL FIGURES.—EQUALITIES.

Definitions.

Def. 22. A triangle is a figure bounded by three straight lines.

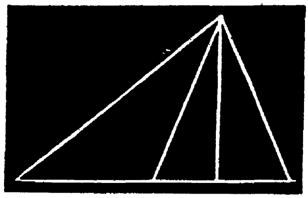


Fig. 72.

The side of a triangle on which it may be supposed to stand is called its base and the opposite corner is then called its vertex.

The line drawn perpendicular to the base from the vertex is called its height or altitude.

The line joining a vertex to the middle point of the opposite side is called a median.

Every triangle has three angles as well as three sides, or six parts altogether. It has also an area.

Def. 23. An equilateral triangle is one which has its three sides equal.

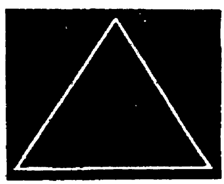


Fig. 73.

Def. 24. An isosceles triangle is one which has two sides equal.

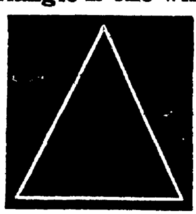


Fig. 74.

Hence an equilateral triangle is also isosceles.

The two equal sides of an isosceles triangle are usually spoken of as its *sides*, and the remaining side as its *base*.

Def. 25. A scalene triangle is one which has three unequal sides.

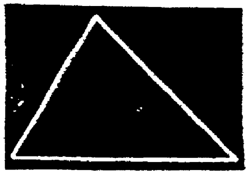


FIG. 75.

Def. 26. A right-angled triangle is one which has a right angle.

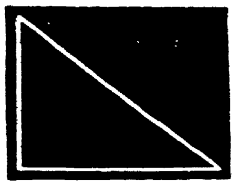


FIG. 76.

In a right-angled triangle the side which is opposite to the right angle is called its hypotenuse, and the other two sides its base and perpendicular.

Def. 27. An obtuse-angled triangle is one which has an obtuse angle.

FIG. 77.

Def. 28. An acute-angled triangle is one which has *three* acute angles.

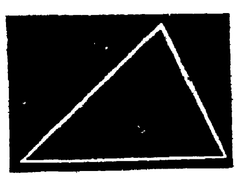


FIG. 78.

Def. 29. A quadrilateral is a plane figure bounded by *four* straight lines.

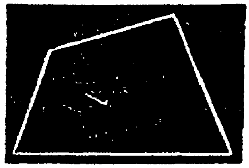


FIG. 79.

Def. 30. A polygon is a plane figure bounded by *four or more* straight lines.

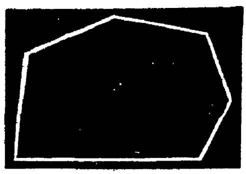


FIG. 80.

Hence a quadrilateral is a polygon.

A five-sided	polygon	is called	a *pentagon*.
A six-sided	„	„	„ *hexagon*.
A seven-sided	„	„	„ *heptagon*.
An eight-sided	„	„	an *octagon*.
A nine-sided	„	„	a *nonagon*.
A ten-sided	„	„	„ *decagon*.
An eleven-sided	„	„	an *undecagon*.
A twelve-sided	„	„	a *dodecagon*.
A fifteen-sided	„	„	„ *quindecagon*.

Def. 31. A convex polygon has each of its angles less than two right angles.

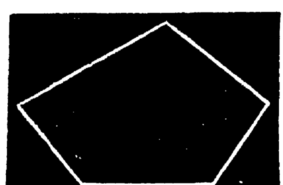


FIG. 81.

Def. 32. A regular polygon has all its sides equal and all its angles equal (see Def. 9).

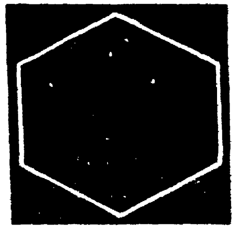


FIG. 82.

Def. 33. The diagonals of a polygon are straight lines joining opposite corners.

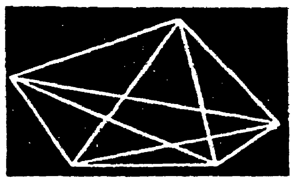


FIG. 83.

Note. If all the sides of a plane rectilineal figure are produced in the sense in which a point would move to trace out the figure, they are said to be produced in order.

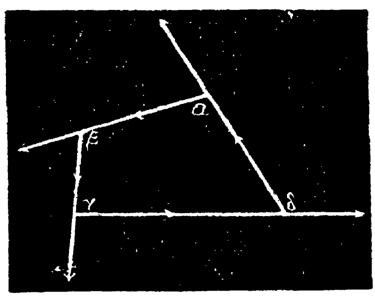


FIG. 84.

The angles formed by producing the sides of a plane rectilineal figure are called its exterior angles.

Of the interior angles of a plane rectilineal figure, those that are non-adjacent to an exterior angle are said to be interior and opposite to that exterior angle. Thus in Fig. 84 the sides are produced in order, and the angles α, β, γ are interior and opposite to the exterior angle δ.

THEOREM 8.

(Euc. I. 32.)

Gen. Enun. *The sum of the angles of a triangle is equal to two right angles.*

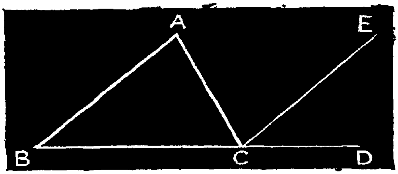


FIG. 85.

Part. Enun. Let ABC be a $\triangle$.

It is reqd. to prove that

$$\angle A + \angle B + \angle C = 2 \text{ rt. } \angle s.$$

Const. Produce BC to D.

Suppose CE to have been drawn $\parallel$ to BA.

Proof. $\because$ CE is $\parallel$ to BA and AC cuts them,

$\therefore \angle A = \text{alt. } \angle ACE$ . . . Th. 6 (A).

And $\because$ CE is $\parallel$ to BA and BD cuts them,

$\therefore \angle B = \text{corresp. } \angle ECD$ . . Th. 6 (B).

$\therefore \angle A + \angle B = \angle ACE + \angle ECD = \angle ACD.$

$\therefore \angle A + \angle B + \angle C = \angle ACD + \angle ACB$

$= 2 \text{ rt. } \angle s.$ . . . Th. 1.

Q.E.D.

Cor. 1. *Any two angles of a triangle are together less than two right angles.* (Euc. I. 17.)

Cor. 2. *If two triangles have two angles of the one equal to two angles of the other, their remaining angles are equal.*

Cor. 3. *If one side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.* (Euc. I. 32.)

Cor. 4. *If one side of a triangle is produced, the exterior angle so formed is greater than either of the interior opposite angles.* (Euc. I. 16.)

QUESTIONS FOR EXAMINATION.—XIII.

1. Classify triangles (i) with regard to their sides, (ii) with regard to their angles.

2. What is a hypothetical construction? Is any use made of a hypothetical construction in the proof of Theorem 8? If so, where?

3. Suppose DAE to have been drawn parallel to the side BC of a triangle ABC, prove that the three angles of the triangle are equal to two right angles without any further construction.

4. Illustrate the truth of Theorem 8 by turning down the corners of a triangular piece of paper so that they may meet at a point in one of the sides.

5. A man walks round and round a triangular course ABC. Draw a diagram to show the three angles through which he turns. Can you give any reason for supposing that the sum of these three angles is equal to four right angles?

6. If the sum of the three angles turned through by the man in question 5 is equal to four right angles, prove that the sum of the angles of the triangle ABC is equal to two right angles.

Exercises.

55. Show that each $\angle$ of an equiang. $\triangle$ = $\frac{2}{3}$ rt. $\angle$.

56. If 2 $\angle$s of a $\triangle$ are complementary, prove that the remaining $\angle$ is a rt. $\angle$.

57. If 2 $\angle$s of a $\triangle$ are together = remaining $\angle$, prove that the $\triangle$ is rt. $\angle$d.

58. Every rt. $\angle$d $\triangle$ has 2 acute $\angle$s and they are complementary.

59. If any 2 $\angle$s of a $\triangle$ are together $>$ remaining $\angle$, the $\triangle$ is acute $\angle$d.

60. State and prove the converse of Ex. 59.

61. One $\angle$ of a $\triangle$ is half each of the other two. Find its magnitude. (36°.)

62. If 2 $\angle$s of a $\triangle$ are together $<$ remaining $\angle$, the $\triangle$ is obtuse $\angle$d.

63. State and prove the converse of Ex. 62.

64. If 2 $\angle$s of a $\triangle$ are 63° 11′ 52″ and 81° 49′ 23″ respectively, what is the remaining $\angle$? (34° 58′ 45″.)

65. The $\angle$ formed by the bisectors of any 2 $\angle$s of a $\triangle$ is always an obtuse $\angle$.

66. *The sum of the $\angle$s of a quadl. = 4 rt. $\angle$s.* (Bomb. Matric.)

67. At least 2 of the $\angle$s of every $\triangle$ are acute.

68. Two $\angle$s of a $\triangle$ are = one another and each is $\frac{1}{2}$ remaining $\angle$. Find the 3 $\angle$s. (45°, 45°, 90°.)

69. The 3 $\angle$s of a $\triangle$ are α, β, γ. $\alpha = 5\gamma, \beta = 4\gamma$. Find α, β, γ. (90°, 72°, 18°.)

70. The 4 $\angle$s of a quadl. are $\alpha, \beta, \gamma, \delta$. $\alpha = 5\delta, \beta = 4\delta, \gamma = 2\delta$. Find $\alpha, \beta, \gamma, \delta$. (150°, 120°, 60°, 30°.)

71. The base BC of a $\triangle$ ABC is produced to D. BE and CE, bisecting the $\angle$s ABC, ACD respy. meet at E. Prove that $\angle$ BEC $= \frac{1}{2} \angle$ BAC.

72. If the sides of a $\triangle$ are produced in order, two, at least, of the ext. $\angle$s so formed must be obtuse.

73. If one side of a quadl. is produced, the ext. $\angle$ so formed is $<$ the sum of any 2 of the int. opp. $\angle$s.

74. If one side of a $\triangle$ is produced, is the ext. $\angle$ $>$ the int. adj. $\angle$? Draw a fig. and give reasons for your answer. (Punj. Eur. Schls. High.)

75. Could the $\angle$s at the base of any $\triangle$ be either obtuse or rt. $\angle$s? Give a reason. (Beng. Eur. Schls. Mid.)

76. The bisector AD of the $\angle$ BAC of a $\triangle$ ABC meets the side BC in D and BC is produced to E. Prove that $\angle$ ABE + $\angle$ ACE = 2 $\angle$ ADE.

77. *If the opp. $\angle$s of a quadl. are $=$, the opp. sides are $\parallel$.*

78. ABCD is a quadl., and the bisectors of the $\angle$s A and B meet at E. Prove that $\angle$ C + $\angle$ D = 2 $\angle$ E.

79. Prove that the sum of the $\angle$s of a $\triangle$ = 2 rt. $\angle$s by joining the vertex to any pt. in the base.

80. From any pt. in the base of a $\triangle$ $\perp$s are drawn to the 2 sides of the $\triangle$. Prove that the sum of the $\angle$s made by the $\perp$s with the base = vert. $\angle$ of $\triangle$. (Punj. Mat.)

81. *The $\perp$ drawn from the rt. $\angle$ of a rt. $\angle$d $\triangle$ to the opp. side divides the rt. $\angle$d $\triangle$ into 2 $\triangle$s, such that the $\angle$s of the first taken in order are equal respy. to the $\angle$s of the second, taken in order.*

*82. The sides of a quadl. are produced in order. Prove that the bisectors of the 4 ext. $\angle$s form another quadl. whose opp. $\angle$s are supplementary.

*83. The bisectors of the $\angle$s B, C of a $\triangle$ ABC meet in D. Prove that

$$\angle \text{BDC} - \tfrac{1}{2}\angle \text{A} = 90^\circ.$$

*84. In a $\triangle$ ABC, $\angle$ B + $\angle$ C = 126° and $\angle$ B − $\angle$ C = 74°. Find $\angle$ A, $\angle$ B and $\angle$ C. (54°, 100°, 26°.)

*85. *Prove that if the sides of any $\triangle$ are produced beyond the base, and the ext. $\angle$s thus formed are bisected, the bisectors will include an $\angle$ = half the sum of the base angles.*

*86. The base BC of a $\triangle$ ABC is produced both ways and the ext. $\angle$s so formed are bisected by st. lines meeting at D. Prove that

$$\angle \text{BDC} + \tfrac{1}{2}\angle \text{A} = 90^\circ.$$

*87. ABC is a $\triangle$. AD, BE, CF are drawn within the $\triangle$ so that $\angle$ BAD = $\angle$ CBE = $\angle$ ACF. If AD, BE, CF do not meet in a pt. prove that they form a $\triangle$ whose $\angle$s, taken in order, are equal respy. to the $\angle$s of the $\triangle$ ABC taken in order.

*88. If a trisector of an ext. $\angle$ of a $\triangle$ is $\parallel$ to a trisector of an int. $\angle$, prove that the other trisector of the ext. $\angle$ is $\parallel$ to a trisector of an int. $\angle$. (Bombay Previous.)

THEOREM 9.

(Euc. I. 32, Cor.)

Gen. Enun. *If the sides of a convex polygon are produced in order the sum of the angles so formed is equal to four right angles.*

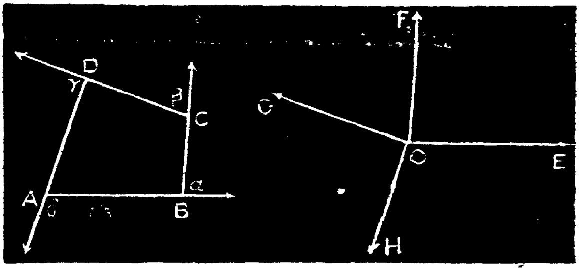


FIG. 86.

Part. Enun. Let ABCD be a convex polygon whose sides are produced in order so as to form the ext. $\angle$s α, β, γ, δ.

It is reqd. to prove that

$$\angle\alpha + \angle\beta + \angle\gamma + \angle\delta = 4 \text{ rt. } \angle\text{s}.$$

Const. Through any point O suppose the st. lines OE, OF, OG, OH to have been drawn ‖ to and in the same sense as AB, BC, CD, DA respy.

Proof. Since OE, OF are drawn respy. ‖ to and in the same sense as the arms of the $\angle\alpha$.

$$\therefore \angle\alpha = \angle\text{EOF} \quad . \quad . \quad . \quad \text{Th. 6, Cor.}$$
$$\text{similarly } \angle\beta = \angle\text{FOG},$$
$$\angle\gamma = \angle\text{GOH},$$
$$\angle\delta = \angle\text{HOE},$$
$$\therefore \angle\alpha + \angle\beta + \angle\gamma + \angle\delta = \angle\text{EOF} + \angle\text{FOG} + \angle\text{GOH} + \angle\text{HOE},$$
$$= 4 \text{ rt. } \angle\text{s} \quad . \quad . \quad . \quad \text{Th. 1, Cor. 2.}$$

Q.E.D.

Cor. *The sum of the interior angles of any convex polygon together with four right angles is equal to twice as many right angles as the polygon has sides.*

Proof. Produce the sides in order, then,

All int. $\angle$s + all ext. $\angle$s = twice as many rt. $\angle$s as the polygon has sides . . . . Th. 1.

$\therefore$ All int. $\angle$s + 4 rt. $\angle$s = twice as many rt. $\angle$s as the polygon has sides . . . . Th. 9.

Q.E.D.

QUESTIONS FOR EXAMINATION.—XIV.

1. Explain the following terms: vertex of a triangle, hypotenuse, diagonal.

2. Draw a diagram of a polygon that is, and another of a polygon that is not, convex.

3. What do you mean by producing the sides of a polygon *in order*?

4. The sides of a polygon can be produced in order first in one sense and then in the opposite sense. Show that two different sets of exterior angles are thus formed.

5. Illustrate the truth of Theorem 9 by rotation about the angular points of the polygon.

6. Prove the corollary of Theorem 9 for a polygon whether convex or not by joining one angular point to all the rest.

Exercises.

89. Show that an ext. $\angle$ of a reg. octagon $= 45°$.

90. How many degs. are there in each ext. $\angle$ of (1) a reg. pentagon, (2) a reg. nonagon? (72°, 40°.)

91. Each ext. $\angle$ of a reg. polygon $= \frac{2}{5}$ of a rt. $\angle$. Find the number of sides. (Madras Matric.) (10.)

92. Each ext. $\angle$ of a reg. polygon is half a degree. Find the number of sides. (720.)

93. How many degs. are there in the sum of the int. $\angle$s of a polygon of (1) 4 sides, (2) 7 sides? (360°, 900°.)

94. *If the sum of the int. $\angle$s of a polygon $=$ 12 rt. $\angle$s, how many sides has it?* (8.)

95. Show that each $\angle$ of a reg. polygon with 15 sides $= \frac{26}{15}$ of a rt. $\angle$. (Punj. Eur. Schls. High.)

96. How many degs. are there in each int. $\angle$ of (1) a reg. hexagon, (2) a reg. dodecagon? (U. P. Eur. Schls. High.) (120°, 150°.)

97. If each int. $\angle$ of an equiangular polygon is 150°, find the number of its sides. (Punj. Mat.) (12.)

98. Six of the int. $\angle$s of a reg. polygon $=$ 9 rt. $\angle$s. Find the number of sides. (8.)

99. Could (1) reg. octagons, (2) reg. hexagons, be fitted together so as to form a perfect mosaic? (No; yes.)

100. Prove that the sum of the int. $\angle$s of a convex octagon is three times the sum of the ext. $\angle$s.

101. How many diagonals has (1) a pentagon, (2) a hexagon? (5, 9.)

102. Draw a pentagon having all its $\angle$s but not all its sides $=$.

103. Draw a pentagon having all its sides but not all its $\angle$s $=$

104. Five of the $\angle$s of a hexagon are 132°, 175°, 151°, 96°, 130°. Find the remaining $\angle$. (36°.)

105. *Each of the $\angle$s of an equiangular polygon of n sides* $= \dfrac{2n-4}{n}$ *rt.* $\angle$s. (U. P. Eur. Schls. High.)

106. Find the ratio of an $\angle$ of a reg. octagon to an $\angle$ of a reg. hexagon. (9 : 8.)

107. How many sides has a convex polygon whose ext. $\angle$s taken in order are together = sum of its int. $\angle$s? (4.)

108. The sides of a reg. octagon are produced both ways till they meet. Show that eight rt. $\angle$s are formed where they meet.

109. A reg. polygon of more than 4 sides must have all its ext. $\angle$s acute.

110. A reg. polygon of more than 4 sides must have all its int. $\angle$s obtuse.

*111. Can any reg. polygon have its ext. $\angle$s (i) 15°, (ii) 20°, (iii) 25°?

*112. Can any reg. polygon have its int. $\angle$s (i) 130°, (ii) 135°, (iii) 140°?

*113. What reg. rect. figs., having the same number of sides, can be fitted together so as to make perfect mosaics?

*114. If the sides of a convex polygon of n sides be produced both ways the sum of the $\angle$s between each alternate pair $= 2(n-4)$ rt. $\angle$s. (Calc. F. E.)

Def. 34. Figures which can be made by superposition to coincide or fit exactly are said to be congruent.

Hence, by Axiom 9, congruent figures are *equal in all respects* or *identically equal.*

The only ultimate test of the identical equality of two figures is that they can be made to coincide by superposition.

Of two congruent figures one is sometimes called the duplicate of the other.

The sign $\equiv$ denotes the congruence of two figures, thus

$$\triangle ABC \equiv \triangle DEF$$

means that the triangles ABC and DEF are equal in all respects, or all the parts of the triangle ABC are equal to all the parts of the triangle DEF, each to each.

When all the angles of one figure are equal to all the angles of another figure, each to each, taken in order, we speak of the figures being *equiangular.* Thus two congruent triangles are equiangular. Notice that we used the word "equiangular" in a different sense on page 35.

Def. 35. Three or more lines are said to be concurrent when they meet in the same point.

Def. 36. Three or more points are said to be collinear when they lie on the same straight line.

THEOREM 10.

(EUC. I. 4.)

Gen. Enun. *If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles contained by those sides equal, the triangles are congruent.*

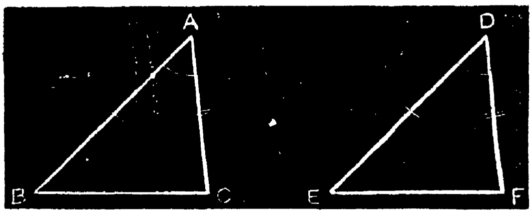


FIG. 87.

Part. Enun. Let ABC, DEF be 2 △s having

$$\begin{cases} AB = DE \\ AC = DF \\ \text{contained} \angle A = \text{contained} \angle D. \end{cases}$$

It is reqd. to prove that

$$\triangle ABC \equiv \triangle DEF.$$

Proof. Apply △ ABC to △ DEF so that A falls on D and AB lies along DE.

Then ∵ AB = DE . . . . Hyp.
∴ B falls on E.
And ∵ ∠ A = ∠ D . . . . Hyp.
∴ AC lies along DF.
And ∵ AC = DF . . . . Hyp.
∴ C falls on F.
∴ △ ABC coincides with △ DEF.
∴ △ ABC ≡ △ DEF. Q.E.D.

QUESTIONS FOR EXAMINATION.—XV.

1. Define congruent figures.
2. Repeat the axiom by which we know that congruent figures are equal in all respects.
3. Which of the axioms does the process of superposition involve?
4. If two figures are congruent must they be equal in area, and if they are equal in area must they be congruent?

5. Illustrate the two different meanings of the term "equiangular" in geometry.

6. Prove Theorem 10 by applying the triangle DEF to the triangle ABC in Fig. 87.

Exercises.

Note.—When two triangles, which have to be proved congruent, overlap each other, it is advisable for beginners to make a separate sketch of each.

115. The line that bisects one of the ∟s of an equilat. △ divides the △ into 2 = parts.

116. *ABCD is a reg. quadl. Prove that the diag. AC bisects each of the ∟s DAB, DCB.*

117. *ABCD is a reg. quadl. Prove that the 2 diags. are =.*

118. ABCD is a reg. quadl. E, F, G are the mid. pts. of AB, BC, CD. Prove that EF = FG.

119. In the fig. of Ex. 118 prove that AF = DF.

120. In the fig. of Ex. 118 prove that BG = DF.

121. *The line that bisects the vert. ∟ of an isos. △ bisects the base and is ⊥ to it.*

122. If the st. line joining the mid. pt. of the base of a △ to the vertex is ⊥ to the base, prove that the △ is isos.

123. AOB, COD are 2 diams. of the same ⊙. Prove that AC = BD.

124. A radius OC of a ⊙ is ⊥ to a diam. AOB. Prove that AC = CB.

125. AOB, COD are 2 diams. of the same ⊙ at rt. ∟s to one another. Prove that AC = CB = BD = DA.

126. *If the diags. of a quadl. bisect each other at rt. ∟s, prove that the quadl. is equilat.*

127. AB bisects CD at rt. ∟s. Prove that any pt. on AB is equidistant from C and D.

128. Prove that the 2 diags. of a reg. pentagon drawn from any angular pt. are =.

129. Prove that 3 of the diags. of a reg. hexagon form an equilat. △.

130. PQ and RS bisect one another at T. Prove that the △ PST can be applied to the △ RTQ so as to coincide with it.

131. ABCD is a quadl. AB = AD and the diag. AC bisects ∟ BAD. Prove that CB = CD and AC bisects ∟ BCD.

132. ABCD is a reg. quadl. If = parts AE, AF are cut off from AB, AD, prove that DE = BF.

133. If 2 △s are congruent, prove that the st. lines joining their vertices to the mid-pts. of their bases are =.

134. *Any pt. on the bisector of the vert.* $\angle$ *of an isos.* $\triangle$ *is equidistant from the extremities of the base.*

135. Two pts. P, Q are taken on the sides of a $\triangle$ equidistant from the vertex, and similarly 2 other pts. R, S. Prove that PS = RQ.

136. ABC is an isos. $\triangle$. The = sides AB, AC are produced to D, E so that AD = AE. Prove that BE = CD.

137. In the quadl. ABCD, if AD = BC and $\angle$ DAB = $\angle$ CBA, prove that DB = CA.

138. Prove that any pt. in a diag. of a reg. quadl. is equidistant from 2 angular pts.

139. ABCD is a quadl. having AD = BC and $\angle$ ADC = $\angle$ BCD. If E is the mid. pt. of DC prove that AE = BE.

140. On the = sides AB, AC of an isos. $\triangle$ ABC, 2 pts. D, E are taken such that AD = AE. Prove that $\triangle$ ADC $\equiv$ $\triangle$ AEB.

141. AB *and* CD *bisect one another. Prove that* AC *is* || *to* DB *and* AD *is* || *to* CB.

142. ABC *is a* $\triangle$. D, E *are the mid. pts. of* BC, BA. *The* $\perp$*s at* D, E *meet at* F. *Prove that* FA = FB = FC.

143. If the opp. sides of a quadl. are = and also one pair of opp. $\angle$s, prove that its opp. sides are ||.

144. If the $\perp$ from each of the angular pts. of a $\triangle$ upon the opp. side always bisects that side, prove that the $\triangle$ is equilat.

145. Prove that if one diag. of a quadl. bisects the other at rt. $\angle$s it divides the quadl. into 2 $\triangle$s which are = in all respects. (Mad. Matric.)

146. ABC is a $\triangle$. BA, CA are produced to D, E so that AD = AB and AE = AC. Prove that DE is || to BC. (Bomb. Matric.)

147. ABC is a $\triangle$. D, E are the mid. pts. of AB, AC. BE, CD are produced to F, G so that EF = BE and DG = CD. Prove that AG and AF are in one and the same st. line. (Bomb. Matric.)

148. *If 2* $\triangle$*s have 2* $\angle$*s of the one = 2* $\angle$*s of the other, each to each, and also the sides adj. to these* $\angle$*s =, prove by superposition that the* $\triangle$*s are congruent.*

149. *Prove that the* $\angle$*s at the base of an isos.* $\triangle$ *are = by supposing the bisector of the vert.* $\angle$ *to have been drawn.*

150. ABCD, EFGH are 2 quadls. having AB = EF, BC = FG, CD = GH, $\angle$ ABC = $\angle$ EFG, $\angle$ BCD = $\angle$ FGH. Prove by superposition that quadl. ABCD $\equiv$ quadl. EFGH.

*151. *The st. lines that join the extremities of = and || st. lines towards the same parts are themselves = and* ||. (Euc. I. 33.)

*152. From P, any pt. within a rt. $\angle$ AOB, PM is drawn $\perp$ to AO and produced to Q so that MQ = PM, and PN is drawn $\perp$ to BO and produced to R so that NR = PN. Prove that QR passes through O. (Bom. Previous.)

THEOREM 11.

(Euc. I. 26.)

Gen. Enun. *If two triangles have two angles of the one equal to two angles of the other, each to each, and also one side of the one equal to the corresponding side of the other, the triangles are congruent.*

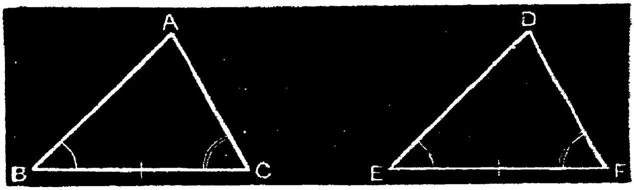


FIG. 88.

Part. Enun. Let ABC, DEF be 2△s having

$$\begin{cases} BC = EF \\ \angle B = \angle E \\ \angle C = \angle F \end{cases}$$

It is reqd. to prove that

$$\triangle ABC \equiv \triangle DEF.$$

Proof. Apply △ ABC to △ DEF so that B falls on E and BC lies along EF.

Then ∵ BC = EF . . . . Hyp.
∴ C falls on F,
and ∵ ∠ B = ∠ E . . . . Hyp.
∴ BA lies along ED,
and ∵ ∠ C = ∠ F . . . . Hyp.
∴ CA lies along FD,
∴ A falls on both ED and FD,
∴ A falls on D the pt. of intersection of ED and FD,
∴ △ ABC coincides with △ DEF,
∴ △ ABC ≡ △ DEF. Q.E.D.

Note. If we are given ∠ A = ∠ D instead of ∠ C = ∠ F,
then ∵ ∠ B = ∠ E and ∠ A = ∠ D,
∴ ∠ C = ∠ F . . . . . Th. 8, Cor. 2.
and the proof may be continued as before.

QUESTIONS FOR EXAMINATION.—XVI.

1. Prove Theorem 11 when BC is given equal to EF, the angle B equal to the angle E, and the angle A equal to the angle D in Fig. 88.

2. If a side and two angles of one triangle are equal to a side and two angles of another triangle, show that the triangles need not be equal in all respects.

3. Deduce from Theorem 11 that a triangle is determined having given a side and the two adjacent angles.

4. What seven separate statements are included in the expression

$$\triangle ABC \equiv \triangle DEF?$$

5. Prove Theorem 11 by *reductio ad absurdum*. (Euclid's proof.)
6. From the defin. of a surface prove that the intersection of two surfaces is a line or lines.

Exercises.

153. *If the bisector of an ∟ of a △ is also ⊥ to the opp. side, the △ is isos.* (Beng. Eur. Schls. Mid.)

154. *Rt. ∟d △s having an acute ∟ of the one = an acute ∟ of the other, and also their hypotenuses =, are equal in all respects.*

155. If △ ABC ≡ △ DEF, prove that the ⊥s from the vertex to the base of each △ are =.

156. △ ABC ≡ △ DEF. If the bisectors of the ∟s A, D meet the bases in G, H respy., prove that AG = DH.

157. PV is the bisector of an ∟ QPR and SVT is drawn ⊥ to meeting PQ, PR in S, T respy. Prove that △ PSV ≡ △ PTV.

158. *Through* R *the mid. pt. of a st. line* PQ *any st. line is drawn and* ⊥*s* PS, QT *are dropped upon it from* P, Q; *show that* PS = QT. (U. P. Eur. Schls. Mid.)

159. *Through* R *the mid. pt. of a st. line* PQ *any st. line is drawn which meets, in* S, T *respy., the lines* PS, QT *drawn* ⊥s to PQ *at* P, Q. *Show that* PS = QT.

160. ABC *is a* △. BDE *is drawn* ⊥ *to the bisector of the* ∟ A, *meeting the bisector in* D *and* AC *in* E. *Show that* BD = DE.

161. ABCD is a quadl. The diag. AC bisects the ∟s A and C. Prove that △ ADC ≡ △ ABC.

162. In the fig. of Ex. 161 prove that AC bisects BD at rt. ∟s.

163. *If the opp. sides of a quadl. are* ∥ *prove that they are also* =.

164. *If the opp. sides of a quadl. are* ∥ *prove that its diags. bisect one another.*

165. ABC is a △. Through A, B, C st. lines are drawn ∥ to BC, CA, AB respy., meeting one another in D, E, F. Prove that A, B, C are the mid. pts. of EF, FD, DE.

166. A is a pt. on the near bank of an impassable river directly opp.

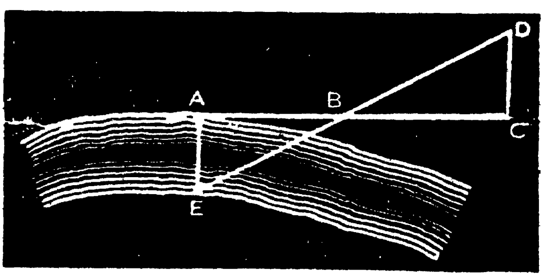


FIG. 89.

an object E on the far bank. AC is drawn ⊥ to AE and bisected at B. CD is drawn ⊥ to AC meeting EB produced at D. Prove that CD = AE.

167. To what useful purpose can the construction in Ex. 166 be turned?

*168. AB and AC are 2 st. lines intersecting at A. D is the mid-pt. of a st. line terminated by AB and AC. Prove that D cannot be the mid-pt. of any other st. line terminated by AB and AC.

*169. *In a △ if a ⊥ be drawn from one extremity of the base to the bisector of the vert.* ∟, *prove that* (i) *it will make with either of the sides containing the vert.* ∟ *an* ∟ = ½ *the sum of the base* ∟*s;* (ii) *it will make with the base an* ∟ = ½ *diffce. of the base* ∟*s.* (Calc. F. E.)

Def. 37. An axis of symmetry of a figure is a line about which the figure can be folded so that one half may coincide with the other half.

A diameter is an axis of symmetry of a circle.

THEOREM 12.

(Euc. I. 5.)

Gen. Enun. *If two sides of a triangle are equal, the angles opposite to these sides are equal.*

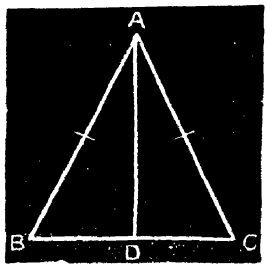


FIG. 90.

Part. Enun. Let ABC be a $\triangle$ having

$$AB = AC.$$

It is reqd. to prove that

$$\angle C = \angle B.$$

Const. Suppose AD to have been drawn bisecting $\angle$ BAC, cutting BC in D.

Proof. In the $\triangle$s ABD, ACD,

$$\because \begin{cases} AB = AC \quad . \quad . \quad . \quad . & \text{Hyp.} \\ AD \text{ is common to both } \triangle\text{s}, \quad . & \\ \angle BAD = \angle CAD \quad . \quad . \quad . & \text{Const.} \end{cases}$$

$$\therefore \triangle ABD \equiv \triangle ACD \quad . \quad . \quad . \quad \text{Th. 10.}$$

$$\therefore \angle C = \angle B. \qquad \text{Q.E.D.}$$

Cor. 1. *If the equal sides of an isosceles triangle are produced, the angles on the other side of the base will be equal.*

Cor. 2. *The bisector of the vertical angle of an isosceles triangle bisects the base.*

Cor. 3. *The bisector of the vertical angle of an isosceles triangle is perpendicular to the base.*

QUESTIONS FOR EXAMINATION.—XVII.

1. What do you understand by an "axis of symmetry" of a figure?
2. Draw an axis of symmetry in (1) an isosceles triangle, (2) an equilateral triangle, and say how many there are in each case.

3. Can you give a reason why only one straight line can be drawn to bisect a given angle?

4. Prove Theorem 12 by folding the triangle about the bisector of the vertical angle.

5. Prove Theorem 12 by showing that the triangle itself and its own "trace" *reversed* are congruent figures.

6. Can you think of a curved surface on which it is possible to draw a straight line? (Ans. Yes; the surface of a round ruler or of a candle extinguisher.)

Exercises.

170. *If a △ is equilateral it is also equiangular.*

171. *The ⊥ from the vertex of an isos. △ to the base divides the △ into 2 = parts.*

172. *The bisector of the vert. ∟ of an isos. △ divides the △ into 2 = parts.*

173. *The st. line joining the vertex to the mid. pt. of the base of an isos. △ divides the △ into 2 = parts.*

174. ABC is an isos. △. D, E, F are the mid. pts. of BC, CA, AB. Prove that DEF is an isos. △.

175. *The st. lines joining the mid. pts. of the sides of an isos. △ to the opp. ends of the base are =.* (U. P. Eur. Schls. High.)

176. ABC is an isos. △. D, E are pts. on the base such that DB = EC. Prove that ∟ ADE = ∟ AED.

177. How many degrees are there in each of the ∟s of an isos. rt. ∟d △? (90°, 45°, 45°.)

178. ABC is an isos. △. The base BC is produced both ways to D, E so that BD = CE. Prove that AD = AE.

179. The = sides AB, AC of an isos. △ ABC are produced beyond the vertex A to E, F so that AE = AF. Prove that EC = FB. (U. P. Eur. Schls. Mid.)

180. ABC, DEF are 2 isos. △s. If vert. ∟ A = vert. ∟ D, prove that ∟ B = ∟ E and ∟ C = ∟ F. (Bomb. Schl. Final.)

181. *If a side of an isos. △ is produced beyond the vertex, prove that the ext. ∟ so formed is double either of the base ∟s of the △.*

182. ABCD is an equilat. quadl. Prove that ∟ A = ∟ C and ∟ B = ∟ D.

183. AB *is a diam. of a* ⊙ *and* C *is a pt. on the circumfce. Prove that* ∟ ACB *is a rt.* ∟.

184. Prove that the st. line drawn ∥ to the base of an isos. △ through the vertex makes = ∟s with the sides of the △.

185. Equilat. $\triangle$s **DBC, ECA, FAB** are described on the sides of an equilat. $\triangle$ **ABC**. Prove that **DEF** is an equilat. $\triangle$.

186. *The diags. of an equilat. quadl. bisect the* $\angle$*s through which they pass.*

187. *The diags. of an equilat. quadl. bisect one another at rt.* $\angle$*s.*

188. Pts. **P, Q, R** are taken on the sides **AB, BC, CA** of an equilat. $\triangle$ **ABC**, so that **AP = BQ = CR**. Prove that **PQ = QR = RP**.

189. *The 3 st. lines joining the angular pts. of an equilat.* $\triangle$ *to the mid. pts. of the opp. sides are =.*

190. The st. lines joining the mid. pts. of the sides of an equilat. $\triangle$ are themselves the sides of an equilat. $\triangle$.

191. **ABC, DBC** are 2 isos. $\triangle$s standing on the same base **BC**. Prove that **AD** or **AD** produced bisects **BC** at rt. $\angle$s. (Mad. Matric.)

192. The = sides **AB, AC** of an isos. $\triangle$ **ABC** are produced to **D, E** so that **AD = AE**. If **BE** and **CD** intersect at **F**, prove that $\triangle$ **BDF** $\equiv \triangle$ **CEF**.

193. **ABC** is an isos. $\triangle$. The bisector of the vert. $\angle$ **A** meets the bisector of the base $\angle$ **B** at **D**. Prove that **CD** bisects the base $\angle$ **C**.

194. In the fig. of Ex. 193 prove that **AD** produced will bisect $\angle$ **BDC**.

195. On the same base and on the same side of it there cannot be 2 isos. $\triangle$s having their vertices outside one another.

*196. The = sides **BA, CA** of an isos. $\triangle$ **BAC** are produced beyond the vertex **A** to pts. **E, F** so that **AE = AF**; **FB, EC** are joined and bisected at **K, L**. Prove that **AK = AL**. (Calc. F. E. Exam.)

*197. *If 2* $\triangle$*s have 2 sides of the one = 2 sides of the other, each to each, and the* $\angle$*s opp. to 2 equal sides =, the* $\angle$*s opp. to the 2 other equal sides are either = or supplementary.* (Calc. F. A.)

*198. If **AC**, the hypotenuse of a rt. $\angle$d $\triangle$ **ABC**, = 2 **AB**, prove that $\angle$ **BAC** = 2 $\angle$ **ACB**.

*199. An equilat. $\triangle$ **A′B′C′** has its angular pts. on the sides of another equilat. $\triangle$ **ABC** so that **A′** falls on **BC**, **B′** on **CA**, and **C′** on **AB**. Prove that the $\triangle$s **B′AC′, C′BA′, A′CB** are congruent. (Bomb. Previous.)

Theorem 13.

(Euc. I. 6.)

Gen. Enun. *If two angles of a triangle are equal, the sides opposite to these angles are equal.*

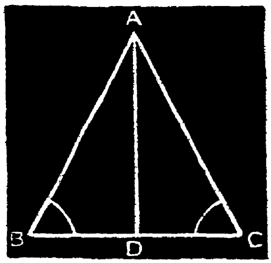


Fig. 91.

Part. Enun. Let **ABC** be a $\triangle$ having

$$\angle B = \angle C.$$

It is reqd. to prove that

$$AC = AB.$$

Const. Suppose **AD** to have been drawn bisecting $\angle$ **BAC**, cutting **BC** in **D**.

Proof. In the $\triangle$s ABD, ACD.

$$\because \begin{cases} \angle B = \angle C & \quad . \quad . \quad . \quad \text{Hyp.} \\ \angle BAD = \angle CAD & \quad . \quad . \quad \text{Const.} \\ AD \text{ is common} & \end{cases}$$

$$\therefore \triangle ABD \equiv \triangle ACD \quad . \quad . \quad \text{Th. 11.}$$

$$\therefore AC = AB. \qquad \text{Q.E.D.}$$

QUESTIONS FOR EXAMINATION.—XVIII.

1. What relation does Theorem 13 bear to Theorem 12? Give the enunciations of two other theorems related in the same way.

2. Prove Theorem 13 by folding the triangle about the bisector of the vertical angle.

3. Prove Theorem 13 by showing that the triangle itself and its own "trace" *reversed* are congruent figures.

4. Prove Theorem 13 on the false assumption that AB is greater than AC. (Euclid's proof.)

5. What do you understand by a side or angle being *common* to two triangles?

6. Which of the axioms is an immediate inference from the definition of a straight line?

Exercises.

200. *If a △ is equiangular it is also equilateral.*

201. ABC is a △ having ∠ B = 2 ∠ A. DB bisecting ∠ B meets AC in D. Prove that BD = DA.

202. If the bisectors of the base ∠s B and C of an isos. △ ABC meet at O, prove that OB = OC.

203. The = sides of an isos. △ ABC are produced and the bisectors of the ∠s at the other side of the base BC meet at O. Prove that OB = OC.

204. In the fig. of Ex. 202 prove that OA bisects ∠ BAC.

205. ABC is a △. D, E are pts. in AC, AB respy. such that BD = CE and ∠ DBC = ∠ ECB. Prove that AB = AC.

206. In the sides AB, AC of an isos. △ ABC, 2 pts. D, E are taken such that DE is ∥ to BC. Prove that AD = AE.

207. *The 2 sides of a △ being produced, if the ∠s on the other side of the base are =, prove that the △ is isos.*

208. State and prove a converse of Ex. 202.

209. The side BC of a △ ABC is produced to D. If the bisector of ∠ ACD is ∥ to AB, prove that CA = CB. (Bomb. Matric.)

210. BC is the base of an isos. △ ABC. BP and CP are drawn ⊥s to AB, AC respy. meeting at P. Prove that BP = CP.

211. *D is a pt. in the hypot.* AC *of a rt.* ∠*d* △ ABC *such that* ∠ DBA = ∠ DAB. *Prove that* DA = DB = DC.

212. *Prove that the hypot.* AC *of a rt.* ∠*d* △ ABC *is double the line joining* B *to the mid-pt. of* AC. (Bom. Schl. Final.)

213. OC is the bisector of an ∠ AOB, and CD is drawn ∥ to AO meeting OB in D. Prove that DO = DC.

214. The sides AB, AC of an isos. △ ABC are produced to D, E respy. so that BD = CE. If BE, CD meet at F, prove that BF = CF.

215. In the fig. of Ex. 214 prove that AF bisects ∠ BAC.

216. Two pts. D, E are taken on the sides AB, AC of an isos. △ ABC so that BD = CE. If BE, CD meet at F, prove that BF = CF.

217. In the fig. of Ex. 216 prove that AF bisects ∠ BAC.

218. In the fig. of Ex. 216 prove that AF produced bisects BC at rt. ∠s.

219. ABC is a △ having AB unequal to AC. Prove by *reductio ad absurdum* that ∠ ABC is unequal to ∠ ACB.

220. State and prove the converse of Ex. 219.

221. The base BC of a $\triangle$ ABC is produced both ways to D and E. If AD = AE and $\angle$ BAD = $\angle$ CAE, prove that AB = AC.

222. In a quadl. ABCD, if $\angle$ A = $\angle$ C and DA = DC, prove that BA = BC.

*223. ABC is an isos. $\triangle$. D is any pt. taken in the side AC. The side AB is produced to E so that BE = CD. Prove that BC bisects DE.

*224. State and prove the converses of Ex. 221.

*225. ABCD is a quadl. having $\angle$ A = $\angle$ B and $\angle$ C = $\angle$ D. Prove that AD = BC.

*226. ABCD is a quadl. having $\angle$ A = $\angle$ B and DC || to AB. Prove that AD = BC.

*227. ABC is an isos. $\triangle$ having $\angle$ B = $\angle$ C = 2 $\angle$ A. If BD bisecting $\angle$ B meets AC in D, prove that AD = BC.

*228. D *is the mid. pt. of the base* BC *of a* $\triangle$ ABC. *If* AD *bisects* $\angle$ BAC *prove that* AB = AC. (Mad. Matric.)

*229. AB, CD are 2 st. lines intersecting at D and the adj. $\angle$s so formed are bisected; if through any pt. X in DC a st. line YXZ is drawn || to AB meeting the bisectors in Y and Z, show that XY = XZ. (Punj. Inter.)

*230. If one acute $\angle$ at the base of a $\triangle$ is double the other $\angle$ at the base and a $\perp$ is drawn from the vertex upon the base, show that the diffce. between the parts into which the base is divided = the smaller side. (Bomb. Matric.)

*231. Perpendiculars AD and BE are drawn from the $\angle$s of a $\triangle$ ABC to meet the opp. sides produced if necessary in D, E; F is the mid. pt. of the side AB; prove that FE = FD. (Calc. F. E.)

*232. ABC is a $\triangle$. D is the mid. pt. of BC and BE, CF are the $\perp$s from B and C on CA, AB. The $\perp$ to EF through E meets the line joining F and D in G. Prove that FD = DG. (Mad. F. A.)

THEOREM 14.

(Philo's proof of Euc. I. 8.)

Gen. Enun. *If two triangles have the three sides of the one equal*

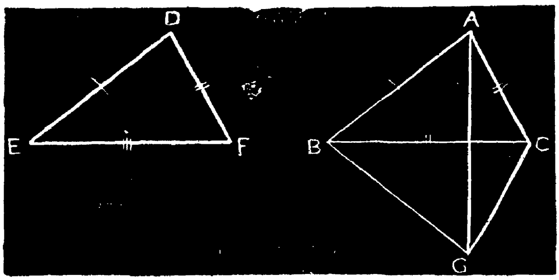


FIG. 92.

to the three sides of the other, each to each the triangles are congruent.

Part. Enun. Let ABC, DEF be 2 $\triangle$s having

$$\begin{cases} AB = DE, \\ AC = DF, \\ BC = EF. \end{cases}$$

It is reqd. to prove that

$$\triangle ABC \equiv \triangle DEF.$$

Const. Let BC be not less than either AB or AC.

Now $\because$ EF = BC, ... Hyp.

$\triangle$ DEF can be placed so that EF coincides with BC and the pt. D falls on the side of BC opp. to A.

Let G be the pt. on which D falls so that GBC is the new position of the $\triangle$ DEF.

Join AG.

Proof. $\because$ AB = DE	Hyp.
= GB	Const.
$\therefore \angle BAG = \angle BGA$	Th. 12.
and $\because$ AC = DF	Hyp.
= GC	Const.
$\therefore \angle CAG = \angle CGA$	Th. 12.
$\therefore \angle BAG + \angle CAG = \angle BGA + \angle CGA$	
$\therefore \angle BAC = \angle BGC$	
$= \angle EDF$	Const.

Hence in the $\triangle$s ABC, DEF

$$\begin{cases} AB = DE \\ AC = DF \\ \text{contained } \angle\ BAC = \text{contained } \angle\ EDF \end{cases}$$

$\therefore \triangle\ ABC \equiv \triangle\ DEF$ . . . . Th. 10.

Q.E.D.

Note 1. The reason for the assumption that BC is not less than either AB or AC is to ensure AG falling between B and C. If BC were less than either AB or AC we should have to place the $\triangle$ DEF so that DE might coincide with AB or DF with AC according as AB or AC were not less than either of the other two sides of the $\triangle$ ABC.

Note 2. The enunciation of Theorem 14 is obtained by interchanging *one of the hypotheses and one of the conclusions in the enunciation* of Theorem 10, thus:—

Theorem 10.	Theorem 14.
Hypotheses.	*Hypotheses.*
AB = DE	AB = DE
AC = DF	AC = DF
$\underline{\angle\ BAC = \angle\ EDF.}$	$\underline{BC = EF.}$
Conclusions.	*Conclusions.*
$\underline{BC = EF}$	$\underline{\angle\ BAC = \angle\ EDF}$
$\angle\ ABC = \angle\ DEF$	$\angle\ ABC = \angle\ DEF$
$\angle\ ACB = \angle\ DFE$	$\angle\ ACB = \angle\ DFE$
$\triangle\ ABC = \triangle\ DEF.$	$\triangle\ BAC = \triangle\ DEF.$

Hence Theorem 14 is a converse of Theorem 10. (See page 49.)

QUESTIONS FOR EXAMINATION.—XIX.

1. In Theorem 14 EF in the figure is made to coincide with BC, because "BC is not less than either AB or AC". Is this condition necessary? If so, why?

2. Prove Theorem 14 when BC in the figure is less than AB and also less than AC.

3. There are three separate hypotheses and four separate conclusions in the enunciation of Theorem 14. State them.

4. What two other theorems have we proved dealing with the congruence of triangles? Give the three separate hypotheses and the four separate conclusions in each case.

5. As far as you know at present what parts of a triangle being given enable you to determine the triangle in all respects?

6. Which of the axioms is an immediate inference from the definition of parallel straight lines?

Exercises.

233. *If the 3 sides of one $\triangle$ are respy. = the 3 sides of another, the 3 $\angle$s of the one = the 3 $\angle$s of the other, each to each.* (Mad. Matric.)

234. Equilat. $\triangle$s on = bases are congruent.

235. A, B, C, D are pts. on the circumfce. of a $\odot$ whose centre is O. If AB = CD prove that $\triangle$ AOB $\equiv$ $\triangle$ COD.

236. Two $\odot$s cut one another at A and B. If their centres are C, D respy. prove that $\triangle$ CAD $\equiv$ $\triangle$ CBD.

237. A and B are pts. on the circumfce. of a $\odot$ whose centre is O. If O is joined to C the mid. pt. of AB, prove that $\triangle$ AOC $\equiv$ $\triangle$ BOC.

238. *If the opp. sides of a quadl. are =, its opp.* $\angle$*s are also =.*

239. In a quadl. ABCD if AB = AD and CB = CD prove that diag. AC divides it into 2 = parts.

240. A pt. D is taken inside an equilat. $\triangle$ ABC so that $\angle$ DBC = $\angle$ DCB. Prove that DA bisects $\angle$ BAC.

241. PQRS is a quadl. in which PQ = SR and PR = SQ. Prove that $\angle$ QPS = $\angle$ RSP, and, if QS, PR meet in O, PO = SO. (Mad. Matric.)

242. The diags. of a quadl. are = and its opp. sides are ||. Prove that all its $\angle$s are rt. $\angle$s.

243. ABC is an isos. $\triangle$. BO, CO bisecting the base $\angle$s meet at O. Prove that AO bisects the vert. $\angle$.

244. On the same base AB and on the same side of it, 2 $\triangle$s ABC, ABD are descd. having AC = BD and BC = AD. If BC and AD cut in E, prove that AE = BE.

245. In the fig. of Ex. 244 prove that $\triangle$ AEC $\equiv$ $\triangle$ BED.

246. *If the opp. sides of a quadl. are = they are also* ||. (U. P. Eur. Schls. Mid.)

247. ABCD is a quadl. AB = CD, AD = BC, AC = BD. Prove that $\angle$ A = $\angle$ B = $\angle$ C = $\angle$ D.

248. BAC *is an* $\angle$. *From centre* A *a* $\odot$ *is descd. cutting* AB, AC *at* D, E *respy. From centres* D, E $\odot$*s are descd. with = radii, cutting one another in* F. *Prove that* AF *bisects* $\angle$ BAC.

249. On the same base and on the same side of it there cannot be 2 different equilat. $\triangle$s constructed. (Bom. Sch. Final.)

250. If $\triangle$s lie on the same side of a common base, and have the sides terminating in one extremity of that base =, the other sides must be unequal. (Mad. Matric.) (Euc. I. 7.)

251. ABCD *is a 4-sided frame loosely jointed at the corners. A cross-bar* AC *will make the frame rigid. Prove this.* (Bom. Schl. Final.)

252. If a $\triangle$ ABC is turned over about the side BC, prove that BC is $\perp$ to the st. line joining the 2 positions of A.

253. ABCD is a 4-sided fig. such that AB = AD and CB = CD. Prove that BD and AC cut at rt. $\angle$s. (Beng. Eur. Schls. Mid.)

254. *If 2* $\odot$*s cut one another, the st. line joining their centres bisects at rt.* $\angle$*s the st. line joining their pts. of intersection.*

255. Four pts. A, B, C, D are so situated that AC = BC and AD = BD. Prove that CD or CD produced bisects AB.

256. **ABCD** is an equilat. quadl. Prove that any pt. on **BD** is equidistant from **A** and **C**.

257. State and prove a converse of Ex. 256.

258. **ABCD** is an equilat. quadl. **AC** is bisected at **O**. If **O** is joined to the angular pts. **B** and **D**, prove that **OB** and **OD** are in one straight line. (Bomb. Matric.)

259. Four pts. are taken in a plane such that the distance between any two is equal to the distance between the other two; find the form of the quadl. obtained by joining the 4 pts. (Bomb. Matric.)

*260. *The 3 $\perp$s drawn from the mid. pts. of the sides of a $\triangle$ are concurrent.* (Punj. Interm.)

*261. *One $\odot$ cannot cut another in more than 2 pts.*

*262. *The 3 $\perp$s drawn from the vertices of a $\triangle$ to the opp. sides are concurrent.*

This point of concurrence is called the **orthocentre** of the $\triangle$.

*263. **ABC** is an isos. $\triangle$. The line **AD** bisecting the base **BC** is produced to **E** and **DE** made equal to **AD**. **E** is joined to the mid. pts. of **AB** and **AC** by lines cutting **BC** in **F** and **G**. Prove that **AFEG** is an equilat. quadl. (Punj. Mat.)

*264. In the equal sides **AB**, **AC** of an isos. $\triangle$ **ABC**, 2 pts. **X**, **Y** are taken so that **AX** = **AY**, and **CX**, **BY** are drawn intersecting at **O**. Prove that **AO** bisects the vert. $\angle$ **BAC**, and also bisects the base at rt. $\angle$s. (Punj. Inter.)

THEOREM 15.

Gen. Enun. *If two right-angled triangles have their hypotenuses equal, and one side of the one equal to one side of the other, the triangles are congruent.*

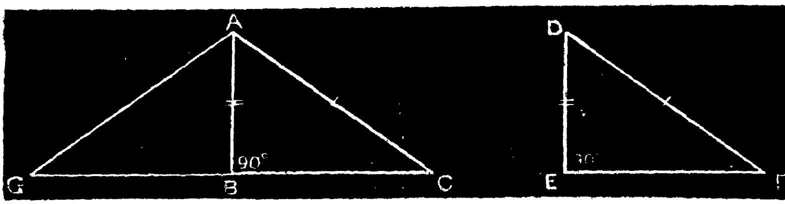


FIG. 93.

Part. Enun. Let **ABC**, DEF be 2 rt. $\angle$d $\triangle$s having

$$\begin{cases} \text{hypot. } AC = \text{hypot. } DF. \\ AB = DE. \end{cases}$$

It is reqd. to prove that $\triangle ABC \equiv \triangle DEF$.

Const. $\because$ DE = AB.

$\triangle$ DEF can be placed so that DE coincides with AB and the pt. F falls on the side of AB opp. to C.

Let G be the pt. on which F falls so that ABG is the new position of the $\triangle$ DEF.

Proof. $\because$ $\angle$s ABC, ABG (= DEF) are rt. $\angle$s	Hyp.
$\therefore$ GBC is a st. line	Th. 2.
$\because$ AG = DF	Const.
= AC	Hyp.
$\therefore$ $\angle$ ACG = $\angle$ AGC	Th. 12.
= $\angle$ DFE	Const.
Hence in the $\triangle$s ABC, DEF	
$\angle$ B = $\angle$ E	Hyp.
$\angle$ C = $\angle$ F	Proved.
AC = DF	Hyp.
$\therefore$ $\triangle ABC \equiv \triangle DEF$	Th. 11.

Q.E.D

QUESTIONS FOR EXAMINATION.—XX.

1. In what five cases does it follow that two triangles are congruent because three parts of one triangle are equal respectively to three parts of the other?

2. In what two cases does it *not* follow that two triangles are congruent because three parts of one triangle are equal respectively to three parts of the other?

3. Prove that the sum and difference of two geometrical magnitudes are together double the greater.

4. Draw diagrams to illustrate adjacent angles, reflex angle, exterior angle.

5. What name is given to the process of reasoning by which we prove a proposition in geometry?

6. What do you mean by saying that two angles are "equal and opposite"?

Exercises.

265. *A, B are 2 pts. on the circumfce. of a ⊙ whose centre is O. Prove that the ⊥ from O on AB divides AB into 2 = parts.*

266. BD, CE are drawn ⊥s to the sides AC, AB of a △ ABC. If BD = CE, prove that AB = AC.

267. A, B, C, D are 4 pts. on the circumfce. of a ⊙ whose centre is O. If the ⊥s OP, OQ from O on AB, CD respy. are =, prove that AB = CD.

268. In the fig. of Ex. 267, if AB = CD prove that OP = OQ.

269. DE, DF are drawn ⊥s to the sides AC, AB of a △ ABC from the mid. pt. of the base. If DE = DF prove that AB = AC.

270. *The ⊥ from the vertex to the base of an isos. △ divides the △ into 2 = parts.*

271. *If 2 △s have 2 sides of the one = 2 sides of the other, each to each, and their heights =, prove that they are = in all respects.*

*272. *The bisectors of the 3 internal ∠s of a △ are concurrent.*

*273. *The bisectors of 2 external ∠s of a △ and the bisector of the 3rd internal ∠ are concurrent.*

*274. *The bisectors of the internal ∠s of a reg. polygon are concurrent.*

*275. *The ⊥s at the mid. pts. of the sides of a reg. polygon are concurrent.*

RECTILINEAL FIGURES.—INEQUALITIES.

THEOREM 16.

(Euc. I. 18.)

Gen. Enun. *If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.*

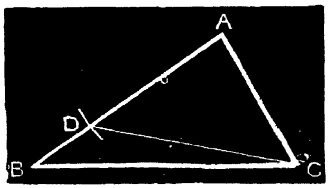


FIG. 94.

Part. Enun. Let ABC be a $\triangle$ having
AB $>$ AC.

It is reqd. to prove that

$\angle$ ACB $>$ $\angle$ ABC.

Const. From AB cut off AD = AC.
Join CD.

Proof. $\because$ AD = AC . . . . Const.
$\therefore$ $\angle$ ADC = $\angle$ ACD . . . . . Th. 12.
but ext. $\angle$ ADC $>$ int. opp. $\angle$ ABC . Th. 8, Cor. 4.
$\therefore$ $\angle$ ACD $>$ $\angle$ ABC
but $\angle$ ACB $>$ $\angle$ ACD
$\therefore$ $\angle$ ACB $>$ $\angle$ ABC Q.E.D.

QUESTIONS FOR EXAMINATION.—XXI.

1. Compare the enunciations of Theorem 16 and Theorem 12 and show, by altering the wording, that one is an extension of the other.

2. By which of the postulates are we able to cut off from the greater of two given straight lines a part equal to the less?

3. Prove Theorem 16 by bisecting $\angle$ BAC in figure 94 by a straight line AE meeting BC in E and joining ED.

4. Prove Theorem 16 by describing a circle with A (in fig. 94) as centre and AC, the shorter side, as radius, cutting BC or BC produced in D and joining AD.

5. Prove Theorem 16 by producing the shorter side AC (in fig. 94) to D and cutting off AD = AB and joining BD.

6. What do you understand by *concurrent lines*, the *medians of a triangle*, the *height of a triangle*?

Exercises.

276. *If the* ∠*s at the base of a* △ *are* = *the* △ *is isos. Prove this indirectly by using Theorem* 16.

277. ABC is a △. If AC < AB prove that ∠ ABC is acute.

278. *The greatest side of any* △ *has the greatest* ∠ *opp. to it.*

279. ABCD is a quadl. having AB = AD but BC < DC. Prove that ∠ ABC > ∠ ADC.

280. ABCD is a quadl. whose opp. sides are ||. If AB > BC, prove that AC lies between DC and the bisector of ∠ BCD.

281. ABCD is a quadl. of which AD is the longest side and BC the shortest. Show that ∠ ABC > ∠ ADC and ∠ BCD > ∠ BAD. (U. P. Eur. Schls. Mid.)

282. The base of an isos. △ is > each of the = sides. Prove that vert. ∠ > 60°.

283. Prove that ∠ A of a △ ABC is an acute ∠, a rt. ∠ or an obtuse ∠ according as the median AD >, = or < ½ BC.

*284. ABCD is a quadl. in which ∠ ABC = ∠ BCD, but CD > AB. Prove that ∠ BAD > ∠ ADC.

*285. *If the vert.* ∠ *of a* △ *is contained by unequal sides, the st. line joining it to the mid. pt. of the base shall fall between the longer side and the st. line that bisects the* ∠. (Bomb. Previous.)

*286. The base of a △ whose sides are unequal is divided into 2 parts by a st. line bisecting the vert. ∠. Prove that the greater part is adjacent to the greater side. (Bomb. Matric.)

THEOREM 17.

(Euc. I. 19.)

Gen. Enun. *If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.*

Part. Enun. Let ABC be a $\triangle$ having

$\angle C > \angle B$.

It is reqd. to prove that

AB > AC.

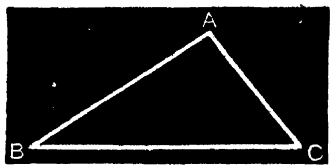


FIG. 95.

Proof. If AB is not > AC

either (i) AB = AC

or (ii) AB < AC.

But if AB = AC

then $\angle C = \angle B$ . . . . . . Th. 12.

which is impossible . . . . . Hyp.

And if AB < AC

then $\angle C < \angle B$ . . . . . . Th. 16.

which is impossible . . . . . Hyp.

$\therefore$ AB must be > AC. Q.E.D.

Note. In Theorem 17 use is made of what is known as the proof **by exhaustion.** It depends upon the principle that when one of several mutually exclusive suppositions must be true, and all these suppositions are proved false *except one*, that remaining supposition must be true.

QUESTIONS FOR EXAMINATION.—XXII.

1. Is Theorem 17 the converse of Theorem 16? If so, why?
2. What method of proof is used in Theorem 17? Describe it.
3. Enunciate a Theorem that we have already proved of which Theorem 17 may be regarded as an extension.

4. In fig. 94 draw CD making $\angle$ BCD = $\frac{1}{2}$($\angle$ C − $\angle$ B) on the same side of BC as A and then give a direct proof of Theorem 17.

5. Deduce the equality of straight angles from Ax. 11.

6. Prove Theorem 8 by the method of rotation.

Exercises.

287. In an obtuse $\angle$d $\triangle$ the greatest side is opp. the obtuse $\angle$.

288. In a rt. $\angle$d $\triangle$ the hypot. is the greatest side.

289. From the vertex A of a $\triangle$ ABC a $\perp$ is drawn meeting the base or base produced in D. Prove that BD < BA and CD < CA. Hence,

290. *Prove that 2 sides of a $\triangle$ are together > the 3rd side.*

291. *The greatest $\angle$ of any $\triangle$ has the greatest side opp. to it.*

292. The bisectors of the base $\angle$s B, C of a $\triangle$ ABC meet at D. If AB > AC prove that DB > DC.

293. The base BC of an isos. $\triangle$ ABC is produced to any pt. D. Prove that AD > AB.

294. *Prove that either side of an isos. $\triangle$ > the st. line drawn from the vertex to any pt. in the base.*

295. Prove that the greater side of a scalene $\triangle$ > the st. line drawn from the vertex to any pt. in the base. (Bomb. Matric.)

296. O is a pt. within the $\triangle$ ABC such that OC = AC. Prove that BC > AC.

297. *Not more than 2 = st. lines can be drawn from a given pt. to a given st. line.*

298. ABC is a $\triangle$ in which BC is not < either AB or AC. Prove that the foot of the $\perp$ from A on BC lies between B and C.

299. The bisector of the $\angle$ A of a $\triangle$ ABC meets BC in D. Prove that AB > BD and AC > CD.

*300. ABC is a $\triangle$ obtuse $\angle$d at A and D, E are pts. in AB, AC respy. Prove that BC > DE.

*301. ABC is a $\triangle$. The bisectors of the ext. $\angle$s at B and C meet at D. If AB < AC, prove that BD > CD.

*302. The = sides AB, AC of an isos. $\triangle$ are produced to D, E respy. BC and DE are produced to meet at F. Prove that AD > AE.

*303. A, B, C are pts. taken in order on the circumfce. of a semicircle. Prove that AC > AB and AC > BC.

*304. The side BA of a $\triangle$ ABC is produced to D and the $\angle$s CAD, CBA are bisected by st. lines meeting in E. BE cuts AC in F. Prove that EF > AF. (Cal. F. E.)

*305. In a rt. $\angle$d $\triangle$ the line joining the rt. $\angle$ to any pt. (except the mid. pt.) of the hypot. > one part of the hypot. and < the other. (Calc. Mat.)

Theorem 18.

(Euc. I. 20.)

Gen. Enun. *Any two sides of a triangle are together greater than the third side.*

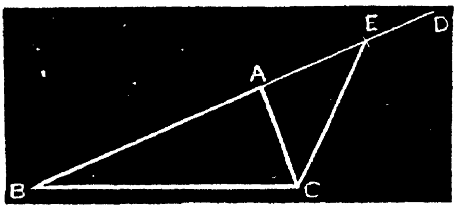


Fig. 96.

Part. Enun. Let ABC be a △.

It is reqd. to prove that
any two sides are together > the third side.

Const. Produce BA to D.
From AD cut off AE = AC.
Join CE.

Proof. ∵ AC = AE . . . . Const.
∴ ∟AEC = ∟ACE . . . . Th. 12.
But ∟BCE > ∟ACE.
∴ ∟BCE > ∟AEC.
> ∟BEC.
∴ BE > BC . . . . . Th. 17.
But BE = BA + AE = BA + AC . . Const.
∴ BA + AC > BC.
Similarly it may be proved that
AB + BC > AC.
AC + CB > AB. Q.E.D.

Cor. *The difference of any two sides of a triangle is less than the third side.*

QUESTIONS FOR EXAMINATION.—XXIII.

1. In Fig. 96 prove that AB + BC > AC.

2. Is it possible to construct a triangle whose sides measure 7 in., 9 in., 16 in. respectively? Give a reason for your answer.

3. From which of the definitions may Theorem 18 be regarded as an immediate inference?

4. Can any two *angles* of a triangle be together greater than the third? Give examples.

5. Prove Theorem 18 by drawing the bisector of an angle of a triangle to meet the opposite side and using Theorem 8, Cor. 4, and Theorem 17.

6. Draw two straight lines equal in magnitude but opposite in sense.

Exercises.

306. *Any 3 sides of a quadl. are together > the 4th side.* (Beng. Eur. Schls. Mid.)

307. ABC is a △ and D is any pt. in AC. Prove that AC + BC > AD + BD.

308. Prove that perim. of pentagon ABCDE > perim. of △ ABD.

309. The pentagon PQRST has its vertices on the sides of another pentagon ABCDE. Prove that perim. of pentagon PQRST < perim. of pentagon ABCDE.

310. Either side of an isos. △ > half the base.

311. ABC is a △. D, E are pts. on AB, AC respy. Prove that perim. of △ ABC > perim. of quadl. BDEC.

312. *Two opp. sides of a quadl. are together < sum of the diags.*

313. O *is any pt. inside a* △ ABC. *Prove that* OA + OB + OC > $\frac{1}{2}$ (AB + BC + CA). (Allah. Mat.)

314. O is any pt. inside a hexagon ABCDEF. Prove that OA + OB + OC + OD + OE + OF > $\frac{1}{2}$ (AB + BC + CD + DE + EF + FA).

315. *The perim. of a quadl. > the sum of the diags.*

316. The sum of the 4 st. lines drawn from a given pt. to the vertices of any quadl. > the sum of the diags. of the quadl. Prove this and mention the exceptional case. (Punj. Inter.)

317. No st. line can be drawn in a ⊙ > a diam.

318. *The diags. of a quadl. are together > half the perimeter.*

319. E is a pt. on the side AB of an isos. △ ABC and D is the mid. pt. of the base. Prove that the diffce. between DE and DB < diffce. between AC and AE.

*320. *Inside a* △ ABC *a pt.* O *is taken. Prove that* AB + AC > OB + OC. (Euc. I. 21.)

*321. *Inside a ⊙ a pt.* P *is taken. Of all st. lines that can be drawn to the circumfce. from* P, *the greatest is that which passes through the centre.*

*322. *Inside a ⊙ a pt.* P *is taken. Of all st. lines that can be drawn to the circumfce. from* P, *the least is that which, when produced, passes through the centre.*

*323. *Outside a ⊙ a pt.* P *is taken. Of all st. lines that can be drawn to the circumfce. from* P, *the greatest is that which passes through the centre.*

*324. *Outside a ⊙ a pt.* P *is taken. Of all st. lines that can be drawn to the circumfce. from* P, *the least is that which, when produced, passes through the centre.*

*325. O *is any pt. within a* △ ABC. *Prove that* AB + BC + CA > OA + OB + OC. (Allah. Mat.)

*326. *In any* △ ABC, AB + AC > *twice the median* AD. (Calc. Mat.)

*327. *In any* △ *the sum of the medians* < *the perim.* (Bomb. Prev.)

*328. O is any pt. within an equilat. △ ABC. Of the 3 st. lines OA, OB, OC prove that any 2 are together > the 3rd.

*329. *If one* △ *can be placed inside another, prove that it must have a smaller perimeter.*

*330. *In any* △ *the perim.* < *twice the sum of the medians.*

*331. In the △ ABC, AB > AC and E is a pt. on the bisector of ∠ A. Prove that AB − AC > EB − EC.

*332. E is a pt. on the bisector of the ext. ∠ at A of a △ ABC. Prove that EB + EC > AB + AC.

Theorem 19.

(Euc. I. 24.)

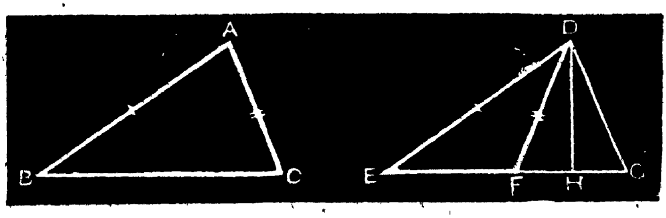
Gen. Enun. *If two triangles have two sides of the one equal to two sides of the other, each to each, but the contained angles unequal, then the base of that which has the greater contained angle is greater than the base of the other.*

Part. Enun. Let ABC, DEF be 2 △s having

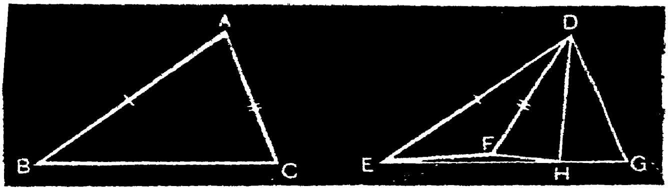
$$\begin{cases} AB = DE \\ AC = DF \\ \text{contained } \angle\ BAC > \text{contained } \angle\ EDF. \end{cases}$$

It is reqd. to prove that

base BC > *base* EF.



Case I.—Fig. 97.



Case II.—Fig. 98.

Const. Apply $\triangle$ ABC to $\triangle$ DEF so that A falls on D and AB lies along DE.

Then $\because$ AB = DE . . . . Hyp.

$\therefore$ B falls on E,

and $\because$ $\angle$ BAC $>$ $\angle$ EDF . . . . . . Hyp.

$\therefore$ AC falls outside $\angle$ EDF.

Let G be the pt. on which C falls so that DEG is the new position of $\triangle$ ABC.

Suppose DH to have been drawn bisecting $\angle$ FDG cutting EG in H.

Join FH.

Proof. Case I. Suppose F is in the st. line EG.

Then EG $>$ EF

i.e., BC $>$ EF. Q.E.D.

Case II. Suppose F is not in the st. line EG.

In the $\triangle$s DHF, DHG

$\because \begin{cases} \text{DF = DG} & \text{. . . Hyp.} \\ \text{DH is common.} & \\ \angle \text{ FDH} = \angle \text{ GDH} & \text{. . Const.} \end{cases}$

$\therefore$ HF = HG . . . Th. 10.

Now in $\triangle$ EFH

EH + HF $>$ EF . . Th. 18.

i.e., EH + HG $>$ EF

i.e., BC $>$ EF. Q.E.D.

QUESTIONS FOR EXAMINATION.—XXIV.

1. In figure 98 suppose AC $>$ AB and prove the proposition.

2. What hypothetical construction is employed in Theorem 19 and which of the postulates?

3. What theorem have we already proved which is a particular case of Theorem 19?

4. How could you illustrate the truth of Theorem 19 with a pair of compasses?

5. How many parts has every triangle and what are they?

6. Prove that straight lines in the same direction make equal angles with every line that cuts them.

Exercises.

333. A, B, C are pts. on the circumfce. of a $\odot$ whose centre is O. If $\angle$ AOB $>$ $\angle$ BOC prove that AB $>$ BC.

334. A, B, C, D are pts. on the circumfce. of a $\odot$ whose centre is O. If $\angle$ AOB $>$ $\angle$ COD prove that AB $>$ CD.

335. AD is a median of the $\triangle$ ABC, and $\angle$ ADB $<$ a rt. $\angle$. Prove that AC $>$ AB.

336. At C the mid. pt. of AB a st. line CD is drawn so that $\angle$ ACD is an obtuse $\angle$. Prove that AD $>$ BD.

337. PQRS is a quadl. having PS = QR and $\angle$ PSR $>$ $\angle$ QRS. Prove that PR $>$ QS.

*338. AB *is a diam. of a* $\odot$. *Of all other st. lines that can be drawn from* A *to pts. on the circumfce. of the* $\odot$, *those which make smaller angles with* AB $>$ *those which make greater angles.*

*339. AB *is a diam. of a* $\odot$ *whose centre is* O *and* P *is a pt. on* AB *between* A *and* O. *Of all other st. lines that can be drawn from* P *to pts. on the circumfce. of the* $\odot$ *those which make smaller angles with* PB $>$ *those which make greater angles.*

*340. ABCD is a quadl. whose opp. sides are $\parallel$ and $\angle$ ABC is obtuse. Prove that AC $>$ BD.

*341. D, E are pts. taken respy. in the sides AB, AC of a $\triangle$ ABC so that BD = CE. If AB $>$ AC prove that BE $>$ DC.

*342. ABC is a $\triangle$ whose sides AB, AC are produced to D, E respy so that BD = CE. If AC $>$ AB prove that BE $>$ DC.

THEOREM 20.

(Euc. I. 25.)

Gen. Enun. *If two triangles have two sides of the one equal to two sides of the other, each to each, but the bases unequal, then the angle contained by the sides of that which has the greater base is greater than the angle contained by the sides of the other.*

Part. Enun. Let ABC, DEF be 2 $\triangle$s having

$$\begin{cases} AB = DE \\ AC = DF \\ \text{base } BC > \text{base } EF. \end{cases}$$

It is reqd. to prove that
contained $\angle$ BAC > *contained* $\angle$ EDF.

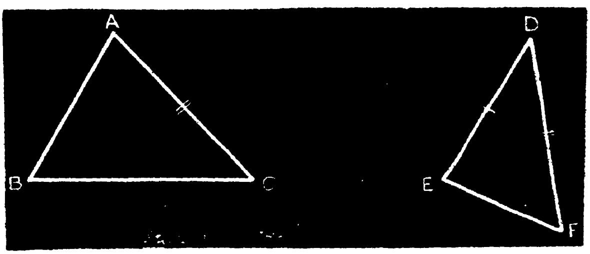


FIG. 99.

Proof. If $\angle$ BAC is not > EDF
either (i) $\angle$ BAC = $\angle$ EDF
or (ii) $\angle$ BAC < $\angle$ EDF.
But if $\angle$ BAC = $\angle$ EDF
then BC = EF . . . . . . Th. 10.
which is impossible. Hyp.
And if $\angle$ BAC < $\angle$ EDF
then BC < EF . . . . . . Th. 19.
which is impossible Hyp.
$\therefore$ $\angle$ BAC must be > $\angle$ EDF. Q.E.D.

QUESTIONS FOR EXAMINATION.—XXV.

1. What resemblance is there between the method of proof used in Theorem 20 and that used in Theorem 17?

2. What relation is there between Theorem 20 and Theorem 19? Enunciate other theorems that we have proved bearing the same relation to one another.

3. What theorem have we already proved which is a particular case of Theorem 20?

4. How could you illustrate the truth of Theorem 20 with a pair of compasses?

5. How can a finite straight line be bisected by folding?

6. How long will it take the minute hand of a clock to turn through (i) 84°, (ii) 168°, (iii) 324°? (Ans. (i) 14 mins., (ii) 28 mins., (iii) 54 mins.)

Exercises.

343. A, B, C are pts. on the circumfce. of a $\odot$ whose centre is O. If AB $>$ BC, prove that $\angle$ AOB $>$ $\angle$ BOC.

344. A, B, C, D are pts. on the circumfce. of a $\odot$ whose centre is O. If AB $>$ CD, prove that $\angle$ AOB $>$ $\angle$ COD.

345. PQRS is a quadl. having PS = QR and PR $>$ QS. Prove that $\angle$ PSR $>$ $\angle$ QRS.

346. PQRS is a quadl. having PS = QR and PQ $<$ RS. Prove that $\angle$ PSQ $<$ $\angle$ SQR.

347. AD is a median of the $\triangle$ ABC and AC $>$ AB. Prove that $\angle$ ADB $<$ a rt. $\angle$. (Calc. F. E.)

348. In the fig. of Ex. 347 if E is any pt. in AD prove that CE $>$ BE.

*349. D, E are pts. taken respy. in the sides AB, AC of a $\triangle$ ABC so that BD = CE. If BE $>$ CD prove that AB $>$ AC.

*350. ABC is a $\triangle$ whose sides AB, AC are produced to D, E respy. so that BD = CE. If BE $>$ DC prove that AC $>$ AB.

*351. PQRS is a quadl. having PS = QR and $\angle$ PSR $>$ $\angle$ QRS Prove that $\angle$ PQR $>$ $\angle$ QPS.

*352. ABC is an isos. $\triangle$ whose vertex is A and D is any pt. within it. If $\angle$ DCB $>$ DBC, prove that $\angle$ BAD $>$ $\angle$ CAD.

Theorem 21.

Gen. Enun. *Of all the straight lines that can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest.*

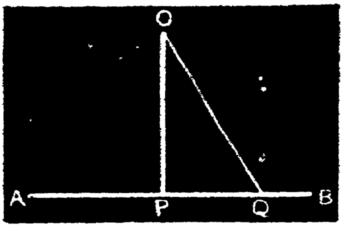


Fig. 100.

Part. Enun. Let AB be a given st. line, O a given pt. outside it, OP the $\perp$ from O to AB and OQ any other st. line meeting AB in Q.

It is reqd. to prove that

$$OP < OQ.$$

Proof.	$\because \angle OPQ = 1$ rt. $\angle$	Hyp.
	And $\angle OPQ + \angle OQP < 2$ rt. $\angle$s .	Th. 8, Cor. 1.
	$\therefore \angle OQP < 1$ rt. $\angle$.	
	$\therefore \angle OQP < \angle OPQ$.	
	$\therefore OP < OQ$	Th. 17.

Q.E.D.

Def. 38. The distance of a point from a straight line is the perpendicular from the point on to the line.

QUESTIONS FOR EXAMINATION.—XXVI.

1. Define the *distance* of a point from a straight line and explain why the distance of a point from a straight line is fixed and constant.

2. Can a straight line be equidistant from two or more points lying on the same side of it?

3. Give an example of a point equidistant from six straight lines.

4. Give an example of a moving point tracing out a line. (Ans. The centre of a wheel rolling along a road.)

5. Give an example of a moving line tracing out a surface. (Ans. The length of a swinging pendulum.)

6. Give an example of a moving surface tracing out a solid. (Ans. The top surface of a pair of bellows at work.)

Exercises.

353. **Prove by Theorem 21 that the hypot. is the greatest side of a rt. $\angle$d $\triangle$.**

354. *In an isos. $\triangle$ the extremities of the base are equidistant from the opp. sides.*

355. *In an isos. $\triangle$ the mid. pt. of the base is equidistant from the opp. sides.*

356. *Any pt. on the bisector of an $\angle$ is equidistant from the arms of the $\angle$.*

357. If a pt. is equidistant from the arms of an $\angle$, it lies on the bisector.

358. The pt. in which the bisector of the vert. $\angle$ of a $\triangle$ meets the base is equidistant from the sides. (Beng. Eur. Schls. Mid.)

359. *The pt. of intersection of the bisectors of 2 $\angle$s of a $\triangle$ is equidistant from the 3 sides of the $\triangle$.*

360. ABC *is a* $\triangle$. AB, AC *are produced to* D, E *respy. The bisectors of the* $\angle$s DBC, ECB *meet at* O. *Prove that* O *is equidistant from* AB, BC, CA.

361. ABC is a $\triangle$. Prove that B and C are equidistant from the st. line joining A to the mid. pt. of BC.

362. C *is the mid. pt. of a st. line* AB. *Prove that* A *and* B *are equidistant from any st. line drawn through* C. (U. P. Eur. Schls. Mid.)

363. If a diag. of a quadl. is equidistant from the outlying angular pts., prove that this diag. bisects the other diag. of the quadl.

364. R is the mid. pt. of PQ. Prove that R is equidistant from || st. lines drawn through P and Q.

365. *If the pt.* E *is halfway between* 2 || *st. lines* AB, CD, *prove that* E *is the mid. pt. of all st. lines through* E *terminated by* AB, CD.

*366. *Of all the st. lines that can be drawn to a given st. line from a given pt. outside it, those that make smaller $\angle$s with the $\perp$ $<$ those which make greater $\angle$s.*

*367. *There cannot be more than 2 pts. of intersection between a st. line and a $\odot$.*

PARALLELOGRAMS AND TRAPEZIUMS.

DEFINITIONS.

Def. 39. A parallelogram is a quadrilateral whose opposite sides are parallel.

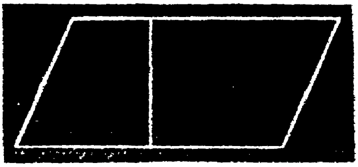


FIG. 101.

The side of a parallelogram on which it may be supposed to stand is called its base. The line drawn perpendicular to the base from any point on the opposite side is called its height or altitude.

Def. 40. A rhombus is a parallelogram having two adjacent sides equal.

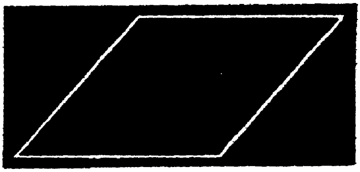


FIG. 102.

Def. 41. A rectangle is a parallelogram having one of its angles a right angle.

FIG. 103.

Def. 42. A square is a rectangle having two adjacent sides equal.

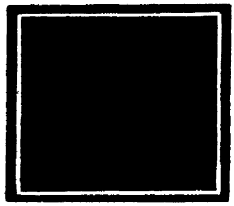


FIG. 104.

Def. 43. A trapezium is a quadrilateral having only one pair of opposite sides parallel.

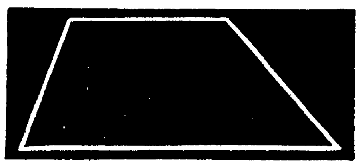


FIG. 105.

Def. 44. An isosceles trapezium is a trapezium in which the sides that are not parallel are equal to one another.

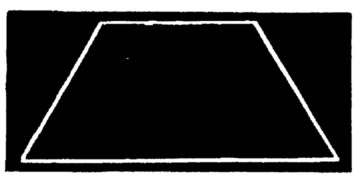


FIG. 106.

Def. 45. The orthogonal projection of one straight line on another of unlimited length is the portion of the latter intercepted between the perpendiculars drawn to it from the extremities of the former.

Thus PQ is the orthogonal projection of AB on CD in Fig. 107 and Fig. 108.

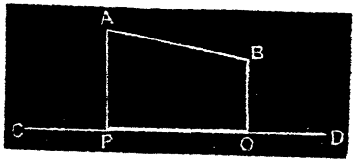


FIG. 107.

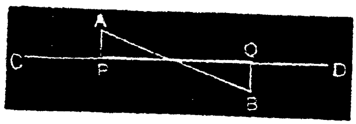


FIG. 108.

THEOREM 22.

(Euc. I. 34.)

Gen. Enun. (A) ***The opposite sides and angles of a parallelogram are equal, and each diagonal bisects the parallelogram.***

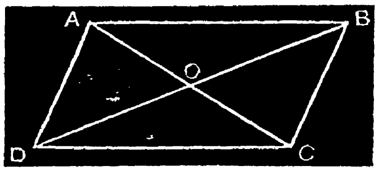


FIG. 109.

Part. Enun. Let ABCD be a $\square$gm and AC, BD its diags.
It is reqd. to prove that

$$\begin{cases} AB = CD \quad ; \quad AD = CB, \\ \angle BAD = \angle DCB \,;\, \angle ABC = \angle CDA, \\ AC,\ BD \textit{ each bisect } \square\text{gm } ABCD. \end{cases}$$

Proof. $\because$ AB is $\parallel$ to DC	Hyp.
and BD cuts them.	
$\therefore \angle ABD =$ alt. $\angle CDB$ . . .	Th. 6 (A).
Again $\because$ AD is $\parallel$ to BC	Hyp.
and BD cuts them.	
$\therefore \angle ADB =$ alt. $\angle CBD$ . . .	Th. 6 (A).

Hence in the $\triangle$s ABD, CDB

$$\begin{cases} \angle ABD = \angle CDB, \\ \angle ADB = \angle CBD, \\ BD \text{ is common.} \end{cases}$$

$\therefore \triangle ABD \equiv \triangle CDB$ . . . . . Th. 11.

so that $\begin{cases} AB = CD\,;\ AD = CB, \\ \angle BAD = \angle DCB, \\ BD \text{ bisects the } \square\text{gm } ABCD. \end{cases}$

Similarly, by using the diag. AC it may be proved that

$$\begin{cases} AB = CD\,;\ AD = CB, \\ \angle ABC = \angle CDA, \\ AC \text{ bisects the } \square\text{gm } ABCD. \end{cases}$$

Q.E.D.

Gen. Enun. (B) *The diagonals of a parallelogram bisect one another.*

Part. Enun. Let ABCD be a ▱gm and let its diags. AC, BD intersect at O.

It is reqd. to prove that

$$\begin{cases} AO = OC, \\ DO = OB. \end{cases}$$

Proof. In the △s AOD, COB

$$\because \begin{cases} \angle ADB = \text{alt.} \angle CBD & \text{Th. 6 (A).} \\ \angle AOD = \text{vert. opp.} \angle COB & \text{Th. 3.} \\ AD = CB & \text{Th. 22 (A).} \end{cases}$$

$$\therefore \triangle AOD \equiv \triangle COB \qquad \text{Th. 11.}$$

$$\text{so that } \begin{cases} AO = OC, \\ DO = OB. \end{cases}$$

Q.E.D.

Cor. 1. *If one pair of adjacent sides of a parallelogram are equal, all its sides are equal.*

Cor. 2. *If one angle of a parallelogram is a right angle, all its angles are right angles.*

Cor. 3. *Parallel straight lines are everywhere equidistant.*

QUESTIONS FOR EXAMINATION.—XXVII.

1. Given a parallelogram how must it be modified to become (*a*) a rectangle, (*b*) a rhombus, (*c*) a square?

2. Explain the term "orthogonal projection".

3. Draw a parallelogram and a triangle on the same base and of the same height.

4. In figure 109, prove that ∠ ABC = ∠ CDA.

5. Through how many degrees does the minute hand of a clock turn in (i) ten minutes, (ii) twenty-five minutes? (Ans. (i) 60°, (ii) 150°.)

6. How could we test the straightness of (1) the edge of a square box, (2) the bore of a gun? Which of the properties of a straight line do we make use of in each case? (Ans. (1) We might apply a straight-edge and see that the two lines do not enclose a space. (2) We might look along it and see that the bore has the same direction throughout, in other words, lies evenly between its two ends.)

Exercises.

368. *All the sides of a sq. are = and all its ∠s are rt. ∠s.*

369. AB and CD are = and ∥ st. lines drawn in the same sense. Prove that AD and BC bisect one another.

370. *A quadl. whose opp. sides are = must be a ▱gm.*

371. *A quadl. whose opp. ∠s are = must be a ▱gm.*

372. *A ▱gm whose diags. are = must be a rect.*

373. If the diag. AC of the ▱gm ABCD bisects ∠ DAB, it must also bisect ∠ DCB.

374. *The diags. of a rect. are =.*

375. *A diag. of a ▱gm is equidistant from the outlying angular pts.*

376. The st. line joining the mid. pts. of 2 opp. sides of a ▱gm is ‖ to the remaining sides.

377. If a ▱gm is not a rect. its diags. are not =.

378. Through the angular pts. of a △ st. lines are drawn ‖ to the opp. sides; prove that the area of the △ so formed is 4 times the area of the original △.

379. Any st. line drawn through the pt. of intersection of the diags. of a ▱gm and terminated by the sides of the ▱gm (*a*) is bisected by each diag., (*b*) divides the ▱gm into 2 = parts. (Bomb. Previous.)

380. *The diags. of a rhombus bisect the ∠s through which they pass.*

381. If the diags. of a quadl. bisect one another it must be a ▱gm.

382. *The diags. of a rhombus cut one another at rt. ∠s.* (U. P. Eur Schls. Mid.)

383. *The diags. of a square are = and cut one another at rt. ∠s.*

384. In the sides AD, BC of a ▱gm ABCD, two pts. E, F are taken so that AE = CF. Prove that AECF is a ▱gm.

385. In the fig. of Ex. 384 prove that EF divides the ▱gm into 2 = parts.

386. *If a rect. and a △ are on the same base and of the same height, the area of the rect. is double that of the △.* (U. P. Eur. Schls. Mid.)

387. *If* AD, BC *are the = sides of an isos. trapezium* ABCD *prove that* ∠ A = ∠ B.

388. In the sides AB, BC, CD, DA of a ▱gm ABCD, 4 pts. P, Q, R, S are taken respy. such that AP = CR and BQ = DS. Prove that PQRS is a ▱gm.

389. The median AD of a △ ABC is produced to E so that DE = AD. E is joined to B and C. Prove that ABEC is a ▱gm.

390. A △ ABC and a ▱gm APQC are on the same base AC and on the same side of it; the ⊥s from B and P on AC are =. Prove that BP and PQ are in one st. line. (Allah. Mat.)

391. If the diag. AC of a ▱gm ABCD bisects ∠ A, prove that the diag. BD bisects ∠s B and D.

392. *The orthogonal projections of = and ‖ st. lines on the same st. line are =.*

393. AB, BC *are respy. = and ‖ to* DE, EF. *Prove that* AC *is = and ‖ to* DF.

394. The st. lines joining the extremities of unequal and ‖ st. lines towards the same parts cannot be ‖.

395. Show by folding that either diag. of a rhombus is an axis of symmetry.

396. How many axes of symmetry has a rectangle, and where do they lie?

397. The opp. ∠s of an isos. trapezium are supplementary. (Bom. Matric.)

*398. ABCD is a ▱gm and AC is produced to E so that CE = AC. If the ▱gm BCEF is completed, prove that the pts. D, C, F are collinear.

*399. The medians BD, CE of a △ ABC are produced to F, G so that BD = DF and CE = EG. Prove that the pts. G, A, F are collinear.

*400. If the bisectors of the ∠s of a quadl. form a rect. the quadl. is a ▱gm.

*401. If 2 pts. P, Q are taken in the = sides AB, AC of an isos. △ ABC so that BP = CQ, show that PQ is ∥ to BC. (Calc. F. E.)

*402. P is any pt. in the base BC of an isos. △ ABC. Prove that the sum of the ⊥s from P to AB and AC is constant.

*403. P is any pt. within an equilat. △. Prove that the sum of the ⊥s from P to the 3 sides is constant.

*404. Prove that a quadl. which has two opp. sides and two opp. obtuse ∠s = is a ▱gm.

*405. ABCD is a ▱gm. P is any pt. on CD. PA, PB are joined and DE, CF are drawn ∥ to PA, PB respy. meeting AB produced in E and F. Prove that length of EF is independent of the position of P. (Beng. Eur. Schls. Mid.)

*406. The longer side of a ▱gm is double the shorter. Prove that the lines bisecting the 4 ∠s will enclose a ▱gm whose diag. = the shorter side of the original ▱gm. (Calc. F. E.)

*407. ABC is any △. The side AC is bisected at D and on AB, BC squares ABEF, BHGC are described (outside △ ABC). Show that EH = 2 BD. (Mad. F. A.)

*408. If from the angular pts. of the squares descd. upon the sides of a rt. ∠d △ ⊥s be let fall upon the hypot. produced, prove that the two extreme ⊥s will together = the hypot. (Bom. Schl. Final.)

*409. ABC is a △ rt. ∠d at C. On BC, AC equilat. △s DBC, ECA are descd., and EA, DB are produced to meet in F. Find the magnitude of ∠ AFB and prove that FC is ⊥ to DE. (Mad. F. A.)

THEOREM 23.

Gen. Enun. *If there are three or more parallel straight lines, and the intercepts made by them on any straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are also equal.*

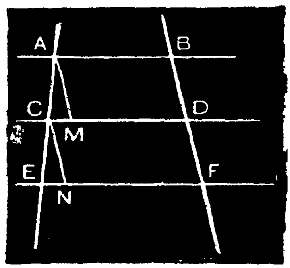


FIG. 110.

Part. Enun. Let AB, CD, EF be $\parallel$ st. lines cut by the st. lines ACE, BDF and let intercept AC = intercept CE.

It is reqd. to prove that

intercept BD = *intercept* DF.

Const. Suppose AM, CN to have been drawn each $\parallel$ to BF, and meeting CD, EF in M, N respy.

Proof. $\because$ CD is $\parallel$ to EF . . . . . . Hyp.
$\therefore \angle$ ACM = corresp. $\angle$ CEN . . Th. 6 (B).
Again $\because$ AM is $\parallel$ to CN . . . . . Th. 7.
$\therefore \angle$ CAM = corresp. $\angle$ ECN . Th. 6 (B).
$\therefore$ in the $\triangle$s ACM, CEN

$$\begin{cases} \angle \text{ACM} = \angle \text{CEN}. \\ \angle \text{CAM} = \angle \text{ECN}. \\ \text{AC} = \text{CE}. \quad . \quad . \quad . \quad . \quad \text{Hyp.} \end{cases}$$

Hence $\triangle$ ACM $\equiv \triangle$ CEN . . . Th. 11.
so that AM = CN.
But AM = BD $\because$ ABDM is a $\square$gm . Th. 22 (A).
And CN = DF $\because$ CDFN is a $\square$gm . Th. 22 (A).
$\therefore$ BD = DF. Q.E.D.

Cor. 1. *The straight line drawn through the middle point of a side of a triangle parallel to the base bisects the remaining side.*

Cor. 2. *The straight line which joins the middle points of the sides of a triangle is parallel to the base.*

QUESTIONS FOR EXAMINATION.—XXVIII.

1. In fig. 110 prove that CM = EN and hence that CD = $\frac{1}{2}$(AB + EF).

2. Give the hypothesis and conclusion of Theorem 28 and enunciate the converse.

3. If the sum and difference of two straight lines are a inches and b inches respectively, find the length of each line in terms of a and b.

4. Prove the corollary of Theorem 9 for a polygon whether convex or not by joining a point within the polygon to its angular points.

5. Explain the terms: *orthocentre*, the *external bisector* of an angle, *intercept*, *perimeter*, *duplicate*.

6. Prove that any triangle can be folded so that its angular points meet at the foot of the perpendicular from an angular point on to the opposite side.

Exercises.

410. AB, CD, EF *are 3 st. lines making = intercepts on any transversal. Prove that* AB, CD, EF *are* ||*s.*

411. All st. lines drawn from the vertex of a △ to pts. in the base are bisected by the st. line joining the mid. pts. of the sides of the △.

412. D, E, F are the mid. pts. of the sides BC, CA, AB of a △ ABC. Prove that BFED, CEFD, AEDF are ▱gms.

413. In the figure of Ex. 412 prove that the △s AFE, BDF, CED, DEF are congruent.

*414. *The st. line joining the mid. pts. of the sides of a △ = $\frac{1}{2}$ base.* (U. P. Eur. Schls. Mid.)

*415. AB, CD *are 2 non-intersecting st. lines. Prove that the sum of the* ⊥*s from* A *and* B *on* CD = *twice the* ⊥ *from the mid. pt. of* AB *on* CD.

*416. AB is a st. line lying outside the ▱gm PQRS. Prove that the sum of the ⊥s from P and R on AB = the sum of the ⊥s from Q and S on AB.

*417. *The st. lines joining the mid. pts. of adj. sides of a quadl. form a ▱gm.* (Bomb. Previous.)

*418. *The st. lines joining the mid. pts. of the opp. sides of a quadl. bisect each other.* (Bomb. Schl. Final.)

*419. The st. lines joining the mid. pts. of 2 opp. sides of a quadl. to the mid. pts. of the diags. form a ▱gm.

*420. The st. lines joining the mid. pts. of opp. sides of a quadl. and the st. line joining the mid. pts. of the diags. are concurrent and bisect one another.

*421. In a quadl. if the diags. are at rt. ∟s, show that the lines bisecting opp. sides are =; if the diags. are =, show that the lines bisecting opp. sides are at rt. ∟s. (Mad. B. C. E.)

*422. ABCD *is a* ▱*gm.* E, F *are the mid. pts. of* AB, CD *respy. Prove that* BF *and* ED *trisect* AC. (Allah. Mat.)

*423. If the △ ABC is rt. ∟d at A, prove that the median through A is half the hypot.

*424. The st. line joining the mid. pts. of the diags. of a trapezium is ∥ to the ∥ sides.

*425. *The st. line joining the mid. pts. of the non-parallel sides of a trapezium = ½ sum of ∥ sides.*

*426. *The medians of a △ are concurrent.* (Calc. F. A.)

This point of concurrence is called the centroid of the △.

*427. *The pt. in which the medians of a △ meet is a pt. of trisection of each median.* (Bomb. Schl. Final.)

*428. P, Q *are the mid. pts. of the sides* AB, AD *of a* ▱*gm* ABCD. *Prove that* CP, CQ *trisect* BD.

*429. If the medians of a △ are =, the △ is equilat.

*430. The sum of the medians of a △ > ¾ perimeter of △. (Mad. F. E.)

*431. From the ends of the base of a △ ⊥s are drawn to the bisector of the vert. ∟; prove that the feet of the ⊥s are equally distant from the mid. pt. of the base. (Calc. Mat.)

*432. In any △ ABC, BP and CQ are ⊥s on any line through A, and M is the mid. pt. of BC. Show that MP = MQ. (Beng. Eur. Schls. Mid.)

*433. If through the mid. pt. of the base of a △ a st. line is drawn ∥ to one of the sides, prove that its intercept between the internal and external bisectors of the vert. ∟ = the other side. (Mad. F. E.)

*434. D, E, F are the mid. pts. of the sides BC, CA, AB of a △ ABC; FG is drawn ∥ to BE meeting DE produced in G; prove that the sides of the △ CFG are equal to the medians of the △ ABC. (Bom. Previous.)

*435. ABC is a △ in which AB = 2 AC. BA is produced to D and ext. ∟ CAD is bisected by AE, cutting BC produced in E; prove that C is the mid. pt. of BE. (Bom. Schl. Final.)

*436. ABC is a △ rt. ∟d at C and D, E, F are the mid. pts. of BC, CA, AB respy. If EF and DF, produced if necessary, meet the ⊥ from C on AB in G and H respy. prove that AG is ∥ to BH. (Mad. F. A.)

LOCI.

Imagine a point moving at a constant distance from a fixed point. Its path will clearly be the circumference of a circle having the fixed point for centre and the constant distance for radius. All points on the circumference of this circle, and no other points, are at the constant distance from the fixed point. This is expressed by saying that the circumference of this circle is the **locus** of a point moving under the given condition.

Similarly we can understand the locus of a point moving at a constant distance from a given straight line to be two straight lines parallel to, but on opposite sides of, the given straight line and at the constant distance from it. Hence :—

Def. 46. If every point on a line, or group of lines, satisfies a given condition, and no other point does so, then that line or group of lines is called the **locus** of the point satisfying that condition.

In proving a required locus it is necessary to show :—

(i) *That every point satisfying the given condition lies on the supposed locus.*

(ii) *That every point on the supposed locus satisfies the given condition.*

If the locus of one point cuts the locus of another point, the point or points where the loci intersect must satisfy the condition of each locus. This suggests a method for finding the position of a point which is subject to two conditions. Suppose, for example, we wish to find a point distant an inch from two given points A and B. If such a point exists at all, it will lie at the intersection of the two circles described from A and B respectively as centres and each of radius one inch, and there will be two solutions, one solution or no solution, according as these circles cut one another, touch one another, or lie entirely outside one another.

The form of a locus or part of a locus can be obtained by finding several consecutive points which satisfy the given condition and joining them by a continuous line. This is called **plotting** the **locus**.

THEOREM 24.

Gen. Enun. *The locus of a point which is equidistant from two fixed points is the perpendicular bisector of the straight line joining the two points.*

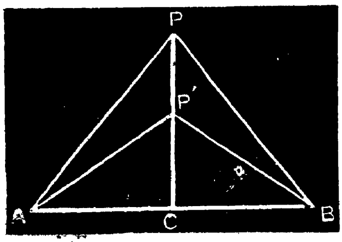


FIG. 111

Part. Enun. (1) Let A, B be two fixed pts.
Let P be any pt. equidistant from A and B.
It is reqd. to prove that
P *lies on the perp. bisector of the st. line joining* A, B.
Const. Suppose AB, the st. line joining A, B, bisected at C.
Join PC, PA, PB.
Proof. In the $\triangle$s PCA, PCB

∵ { CA = CB . . . . Const.
PC is common.
PA = PB . . . . Hyp.

∴ ∟ PCA = ∟ PCB . . . . . . Th. 14.

That is P lies on the perp. bisector of the st. line joining A, B.

Part. Enun. (2) Let A, B be two fixed pts.
Let P′ be any pt. lying on P′C the perp. bisector of the st. line joining A, B.
It is reqd. to prove that
P′ *is equidistant from* A *and* B.
Const. Join P′A, P′B.
Proof. In the $\triangle$s P′CA, P′CB

∵ { CA = CB . . . . Hyp.
P′C is common.
∟ P′CA = ∟ P′CB . . . . . Hyp.

∴ P′A = P′B . . . . Th. 10.

That is P′ is equidistant from A and B.
Hence the reqd. locus is the perp. bisector of AB. Q.E.D.

Theorem 25.

Gen. Enun. ***The locus of a point which is equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.***

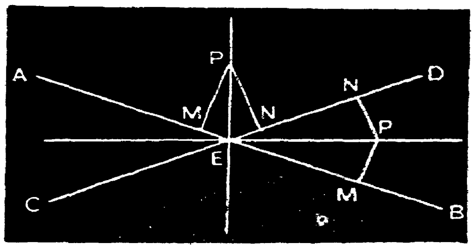


Fig. 112.

Part. Enun. (1) Let AEB, CED be two intersecting st. lines.
Let P be any pt. equidistant from AEB and CED.

It is reqd. to prove that

P *lies on one or other of the bisectors of the* ∟*s between* AEB *and* CED.

Const. Suppose PM, PN drawn ⊥s to AEB, CED respy.
Join PE.

Proof. In the △s PEM, PEN

$\because \begin{cases} \text{PM} = \text{PN} & \text{. . . . . . Hyp.} \\ \text{PE is common.} & \\ \text{rt.} \llcorner \text{PME} = \text{rt.} \llcorner \text{PNE} & \text{. . . Const.} \end{cases}$

$\therefore \llcorner \text{PEM} = \llcorner \text{PEN}$ . . . . . Th. 15.

That is P lies on one or other of the bisectors of the ∟s between AEB and CED.

Part. Enun. (2) Let AEB, CED be two intersecting st. lines (see figure next page).

Let P′ be any pt. lying on P′E, one or other of the bisectors of the ∟s between AEB and CED.

It is reqd. to prove that

P′ *is equidistant from* AEB *and* CED.

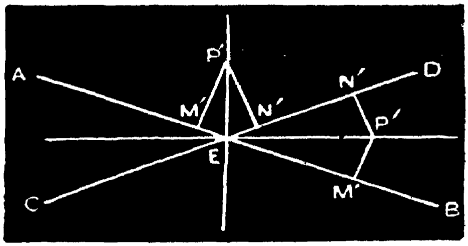


FIG. 112.

Const. Suppose P'M', P'N' drawn ⊥s to AEB, CED respy.

Proof. In the △s P'EM', P'EN'

$\because \begin{cases} \angle P'EM' = \angle P'EN' \quad . \quad . \quad . \quad . & \text{Hyp.} \\ \text{rt.} \angle P'M'E = \text{rt.} \angle P'N'E \quad . \quad . & \text{Const.} \\ P'E \text{ is common.} \end{cases}$

$\therefore P'M' = P'N'$ . . . . Th. 11.

That is P' is equidistant from AEB and CED.

Hence the reqd. locus is the pair of bisectors of the ∠s between AEB and CED.

Q.E.D

QUESTIONS FOR EXAMINATION.—XXIX.

1. Define the locus of a point satisfying a given condition. Give an example.

2. Suppose it proved that all points satisfying a given condition lie on a certain line. Can we conclude that this line is the locus of these points? If not, why not? (Ans. No. We can only conclude that the locus of these points lies along this line. It might consist of only portions of this line.)

3. What is the locus of :—
   (i) The end of a line rotating about its other end?
   (ii) A point on the circumference of a circle rotating about its centre?
   (iii) A point on a radius of a circle rotating about its centre?
   (iv) The centre of a circle rolling along a fixed straight line?
   (v) The centre of a circle rolling along the circumference of a fixed circle?

4. Write down the converse propositions of the proposition: "If the opposite sides of a four-sided figure are parallel its opposite sides are equal and its diagonals bisect one another". Which of them are true?

5. What do you understand by *deductive* reasoning? Give an example.

6. Define the term *transversal*. Name the various angles made by a transversal with two straight lines.

Exercises.

437. The locus of a pt. at a const. distance from a fixed st. line is a pair of lines || to the fixed line.

438. The locus of a pt. equidistant from 2 fixed || st. lines is a st. line || to each of them.

439. The locus of the vertex of a △ on a given base and having the median bisecting the base of a given length is a ⊙.

440. Find the locus of the centre of a ⊙ of given radius which rolls along the inside of the circumfce. of a given ⊙.

441. Find the locus of pts. on the radii of a given ⊙ at equal distances from the centre.

442. Find the locus of all pts. in a given direction from a fixed pt.

443. Find the locus of the vertices of isos. △s on a given base.

*444. Find the locus of mid. pts. of st. lines drawn from a given pt. to meet a given st. line.

*445. St. lines are drawn from a given pt. to meet a given st. line and then produced to double their lengths. Find the locus of their extremities.

*446. A line of given length moves with its ends on 2 fixed st. lines at rt. ∟s to one another. Find the locus of its mid. pt. (Calc. Mat.)

*447. A pt. moves so that the diffce. of its distances from 2 fixed intersecting st. lines is const. Find its locus.

*448. A pt. moves so that the sum of its distances from 2 fixed intersecting st. lines is const. Find its locus.

Intersection of Loci.

449. Find a pt. in a given st. line equidistant from 2 given pts. When is there no such pt.?

450. Find a pt. in a given st. line equidistant from 2 given st. lines. Show that there are generally 2 such pts. When is there only one such pt. and when is there no such pt.? (Mad. Mat.)

451. Find a pt. in a given st. line at a given distance from a given pt. Show that there may be 2 such pts. When is there no such pt.?

452. Find a pt. in a given st. line at a given distance from a given st. line. Show that there are generally 2 such pts. When is there no such pt.?

453. Find a pt. at a given distance from a fixed pt. and equidistant from 2 other fixed pts. Show that there may be 2 such pts. When is there no such pt.?

454. Find a pt. equidistant from 2 given || st. lines and at a given distance from a given pt. Show that there may be 2 such pts. When is there no such pt.?

455. Find a pt. equidistant from 3 pts. which are not collinear.

456. Find a pt. equidistant from the sides of a △. Show that there are 4 such pts.

457. Find a pt. equidistant from 2 given st. lines and at a given distance from a fixed pt. Show that there may be 4 such pts. When are there only 2 such pts. and when is there no such pt.?

458. Find a pt. equidistant from 2 given st. lines and at a given distance from another given st. line. Show that there are generally 4 such pts. When are there only 2 such pts. and when is there no such pt.?

459. Find a pt. equidistant from 2 given st. lines and also equidistant from 2 given pts. Show that there are generally 2 such pts. When is there only one such pt. and when is there no such pt.? (Calc. F. A.)

460. Two men desire to build a well equidistant from each of 2 intersecting straight roads, and also equidistant from their 2 houses on one of the roads. Show how to find the position of the well. (Bomb. Matric.)

*461. Find a pt. at a given distance from 2 given st. lines. Show that there are generally 4 such pts. When is there no such pt.?

*462. Find a pt. equidistant from 2 given pts. and at a given distance from a given st. line. Show that there are generally 2 such pts. When is there no such pt.?

*463. Find a pt. at a given distance from a given pt. and at the same distance from a given st. line. Show that there may be 2 such pts. When is there no such pt.?

Plotting Loci.

In working out Exercises 465 to 471 the student should find some consecutive points from conveniently chosen *numerical* data.

464. Plot the locus of the mid. pts. of st. lines drawn from a fixed pt. to pts. on the circumfce. of a fixed ⊙.

465. Plot the locus of pts. equidistant from a given pt. and a given st. line.

466. Plot the locus of pts. twice the distance from a given pt. as from a given st. line.

467. Plot the locus of pts. twice the distance from one given pt. as from another given pt.

468. Plot the locus of a pt. which moves so that the sum of its distances from 2 given pts. is const.

469. Plot the locus of a pt. which moves so that the diffce. of its distances from 2 given pts. is const.

470. Plot the locus of a pt. which moves so that its distance from one of 2 st. lines intersecting at rt. ∟s is double its distance from the other.

471. Plot the locus of the mid. pt. of a st. line PQ whose extremities lie on 2 intersecting st. lines OX, OY in such a way that OP + OQ = const.

CONCURRENT LINES IN A TRIANGLE.

Note. These theorems are collected here for convenience though each occurs as an exercise in its appropriate place elsewhere.

ADDITIONAL THEOREM I.

Gen. Enun. *The perpendicular bisectors of the sides of a triangle are concurrent.*

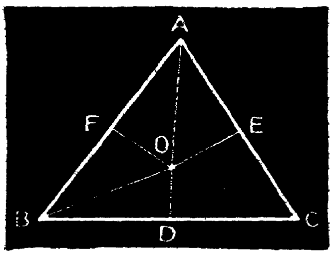


FIG. 112A.

Part. Enun. Let D, E, F be the mid. pts. of the sides BC, CA, AB respy. of a △ ABC. The ⊥s to AC at E and AB at F must intersect at some pt. O. (Why?) Join OD.

It is reqd. to prove that

OD *is* ⊥ *to* BC.

Const. Join AO, BO, CO.

Proof. ∵ △ OFA ≡ △ OFB . . . . . . (Why?)

∴ OA = OB.

Similarly OA = OC

∴ OB = OC

Now in the △s ODB, ODC

$$\because \begin{cases} OB = OC & \text{. . . . . . Proved.} \\ OD \text{ is common} & \\ BD = CD & \text{. . . . . . Hyp.} \end{cases}$$

∴ ∟ODB = ∟ODC . . . . . Th. 14.

Hence OD is ⊥ to BC Q.E.D.

Note. An alternative proof by *loci* is left as an exercise for the student.

Additional Theorem II.

Gen. Enun. *The perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.*

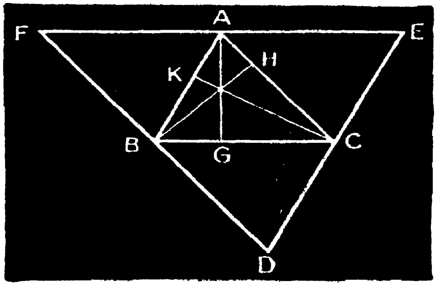


Fig. 112B.

Part. Enun. Let ABC be a △.

It is reqd. to prove that
the ⊥s *drawn from* A, B, C to BC, CA, AB *respy. are concurrent.*

Const. Through A, B, C suppose EF, FD, DE drawn ∥ to CB, AC, BA respy.

Proof. ∵ AFBC and EABC are ▱gms.,

∴ FA = BC = AE . . . . . Th. 22 (A).

That is A is the mid. pt. of FE.

Similarly B, C are the mid. pts. of DF, ED respy.

If, then, AG, BH, CK be drawn ⊥ to EF, FD, DE respy. they will be concurrent . . . Addit. Th. I.

But ∟ AGC = alt. ∟ FAG (a rt. ∟) . . Th. 6 (A).

∴ AG is ⊥ to BC.

Similarly BH, CK are ⊥ to CA, AB respy.

Hence the ⊥s drawn from A, B, C to BC, CA, AB respy. are concurrent. Q.E.D.

Additional Theorem III.

Gen. Enun. *The internal bisectors of the angles of a triangle are concurrent.*

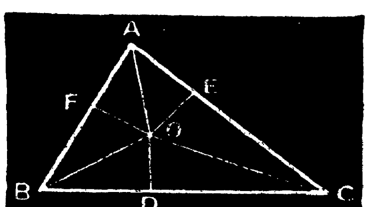


Fig. 112C.

Part. Enun. Let ABC be a $\triangle$. The internal bisectors of $\angle$ B and $\angle$ C must intersect at some pt. O. (Why?) Join OA.

It is reqd. to prove that

$$\angle OAB = \angle OAC$$

Const. Suppose OD, OE, OF drawn $\perp$ to BC, CA, AB respy.

Proof. $\because \triangle ODC \equiv \triangle OEC$ . . . . . (Why?)

$\therefore$ OD = OE.

Similarly OD = OF

$\therefore$ OE = OF

Now in the $\triangle$s OEA, OFA

$\because \begin{cases} OE = OF & \text{. . . . . Proved.} \\ OA \text{ is common} & \\ \text{rt. } \angle OEA = \text{rt. } \angle OFA & \text{. . . . Const.} \end{cases}$

$\therefore \triangle OEA \equiv \triangle OFA$ . . . . . Th. 15.

So that $\angle OAB = \angle OAC$. Q.E.D.

Note. An alternative proof by *loci* is left as an exercise for the student.

ADDITIONAL THEOREM IV.

Gen. Enun. *The medians of a triangle are concurrent.*

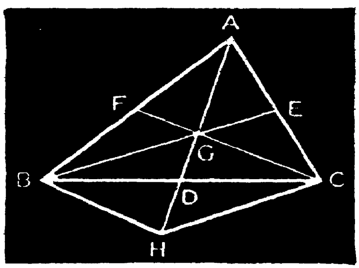


FIG. 112D.

Part. Enun. Let ABC be a $\triangle$. The medians BE and CF must intersect at some pt. G. (Why?) Join AG and produce AG to meet BC in D.

It is reqd. to prove that

$$BD = DC.$$

Const. Produce AD to H making GH = AG.

Join BH, CH.

Proof. $\because$ G is the mid. pt. of AH (Const.) and E is the mid. pt. of AC (Hyp.)

$\therefore$ GE is $\parallel$ to HC . . . . . Th. 23, Cor. 2.

That is BG is $\parallel$ to HC.

Similarly GC is $\parallel$ to BH

$\therefore$ BHCG is a $\square$gm.

But the diagonals of a $\square$gm. bisect one another. Th. 22 (B).

$\therefore$ BD = DC. Q.E.D.

PRACTICAL SECTION.

INTRODUCTORY.

MANY instruments have been devised for constructing geometrical figures, such as the protractor, the set-square and the parallel ruler, each instrument serving some special purpose. But it is found that nearly all the simpler constructions can be effected by means of a ruler (not graduated) and a pair of compasses. Geometers, therefore, have always confined themselves to the use of these two instruments when solving problems in elementary geometry, as is clearly indicated by the postulates laid down at the beginning of the subject.

Every problem in geometry is an exercise in accurate drawing as much as in correct reasoning, and the student should test the accuracy of his results by actual measurement whenever he is able to do so.

Found lines should be thick, given lines fairly thick, construction lines thin, and lines needed only in the proof should be dotted.

No definite rules can be given for solving problems, and herein an exercise in geometry differs from, say, the finding of a cube root in arithmetic or the solving of a quadratic equation in algebra. The intersection of loci suggests the solution of problems of a certain type, but, failing this method, we can often get a useful hint by what is known as the method of analysis.

By this method we assume the construction of the required figure and then, by examining carefully its properties, and noticing how its various parts are related to one another, we can often find a clue to the building up of the finished figure from the *data* of the problem, using only the postulates and constructions already proved. Very much the same process would probably be followed by a man wishing to construct a watch but having little or no previous knowledge of its mechanism. He would take an old watch, pull it to pieces, examine its parts carefully and notice how they had been fitted together, and then reconstruct the watch.

The pulling-to-pieces process is called **analysis**. The putting-together process is called **synthesis**.

The method of analysis can be applied to the proving of theorems as well as to the solving of problems, though not so usefully.

It is sometimes possible to construct two or more figures all satisfying the same set of conditions and yet differing from one another. Such problems, admitting of two or more solutions, are said to be **indeterminate.**

EXAMPLES OF SOLUTION BY ANALYSIS.

Example 1. *Construct an isosceles triangle having given its base and altitude.*

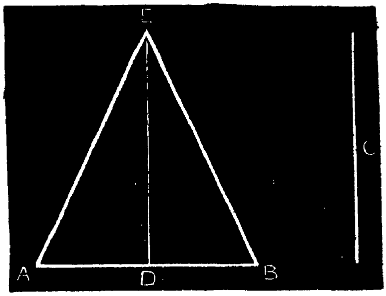


FIG. 118.

Let AB be the given base and C the given altitude.

Analysis. Assume ABE to be the reqd. $\triangle$.
Draw its altitude ED.
$\because$ ED is the altitude of an isos. $\triangle$ . . . Const.
$\therefore$ in $\triangle$s ADE, BDE.
$\begin{cases} \text{AE = BE} \\ \text{DE is common.} \\ \text{rt. } \angle \text{ ADE = rt. } \angle \text{ BDE.} \end{cases}$ . . . . . . Hyp.
$\therefore \triangle$ ADE $\equiv \triangle$ BDE. . . . . Th. 15.
so that DA = DB
and this suggests the

Synthesis. Bisect AB at D . . . . . . Prob. 2.
From D draw DE $\perp$ to AB . . . . Prob. 3.
Make DE = C.
Join AE, BE.
Then shall ABE be the $\triangle$ reqd.

Proof. In $\triangle$s ADE, BDE
$\because \begin{cases} \text{AD = BD} \\ \text{DE is common.} \\ \text{contained } \angle \text{ ADE = contained } \angle \text{ BDE.} \end{cases}$ . . . . . . Const. / Const.
$\therefore$ AE = BE . . . . . Th. 10.
that is, the $\triangle$ is isos.,
and its base and altitude are of the reqd. lengths.

Q.E.F.

· **Example 2.** *Find a point* D *in the base* BC *of a triangle* ABC *such that* AD = ½ (AB + AC).

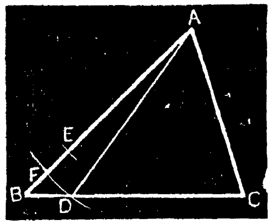


Fig. 114.

Let ABC be the given △.

Analysis. Assume D to be the reqd. pt.
Join DA.
∵ AD = ½ (AB + AC) . . . . . Hyp.
∴ AD = AF if F is the mid. pt. of BE and AE = AC.
And this suggests the

Synthesis. From the greater, AB, of the two sides AB, AC of △ ABC cut off AE = AC.
Bisect BE at F . . . . Prob. 2.
With A as centre and radius = AF desc. a ⊙ cutting BC at D.
Then shall D be the pt. reqd.

Proof. Easy.

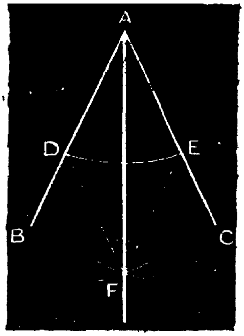


Fig. 115.

PROBLEMS.

LINES AND ANGLES.

Problem I.

(Euc. I. 9.)

Gen. Enun. *To bisect a given angle.*

Part. Enun. Let BAC be a given ∠.

It is reqd. to bisect ∠ BAC.

Const. With A as centre and any radius desc. a ⊙ cutting AB, AC at D, E respy.

With D and E as centres, desc. equal ⊙s such that they intersect at a pt. F within ∠ BAC.

Join AF.

Then shall AF bisect ∠ BAC.

Proof. Join DF, EF.

In △s DAF, EAF,

{DA = EA . . . . Const.
{AF is common.
{FD = FE . . . . Const.

∴ △ DAF ≡ △EAF . . . . . Th. 14.

so that ∠ DAF = ∠ EAF,

that is, AF bisects ∠ BAC. Q.E.F.

Exercises.

472. Divide a given ∠ into 4 = parts.

473. Divide a given obtuse ∠ into 8 = parts.

474. Divide a given ∠ into 2 parts such that one part is ½ of the other.

475. Find the pt. of concurrence of the bisectors of the ∠s of a given △.

476. In a given st. line find a pt. equidistant from 2 given intersecting st. lines.

Problem 2.

(Euc. I. 10.)

Gen. Enun. *To bisect a given finite straight line.*

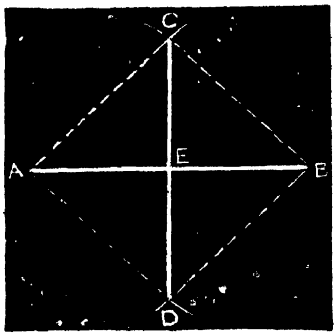


Fig. 116.

Part. Enun. Let AB be a given finite st. line.
It is reqd. to bisect AB.

Const. With A and B as centres, desc. equal ⊙s such that they intersect at C and D.
Join CD cutting AB at E.
Then shall CD bisect AB at E.

Proof. Join CA, CB, DA, DB.
In △s ACD, BCD,

$\because \begin{cases} AC = BC & \text{. . . . . . .} \quad \text{Const.} \\ CD \text{ is common.} & \\ DA = DB & \text{. . . . . . .} \quad \text{Const.} \end{cases}$

∴ △ ACD ≡ △ BCD . . . . Th. 14.
so that ∟ ACD = ∟ BCD.

Again. In △s ACE, BCE,

$\because \begin{cases} AC = BC & \text{. . . . . . .} \quad \text{Const.} \\ CE \text{ is common.} & \\ \text{contained } ∟\, ACE = \text{contained } ∟\, BCE & \text{Proved above.} \end{cases}$

∴ △ ACE ≡ △ BCE . . . . Th. 10.
so, that AE = BE,
that is, CD bisects AB at E. Q.E.F.

Exercises.

477. Divide a given finite st. line into 4 = parts.

478. Divide a given finite st. line into 8 = parts.

479. Divide a given finite st. line into 2 parts such that one part is seven times the other.

480. Find the pt. of concurrence of the perpend. bisectors of the sides of a given △.

481. Find the pt. of concurrence of the medians of a given △.

482. Find a pt. in a given st. line equidistant from 2 given pts. In what case is this impossible? (Punj. Eur. Schls. Mid.)

483. Through a given pt. draw a st. line equidistant from 2 given pts.

484. Through the vertex of a △ draw a st. line equidistant from the ends of the base.

*485. Given the sum and diffce. of 2 st. lines, find their lengths. (Bom. Schl. Final.)

*486. Give the const. only of the particular line in which the vertices of all isos. △s upon a given base FG must lie. (Allah. Ent.)

*487. AB and CD are = st. lines given in position. Show how to find a pt. P such that △s PAB, PCD may be = in all respects. (Mad. F. A.)

*488. Find a pt. in the base of a △ such that its distance from the vertex = half the sum of the sides.

PROBLEM 3.

(EUC. I. 11.)

Gen. Enun. *To draw a straight line perpendicular to a given straight line of unlimited length from a given point in it.*

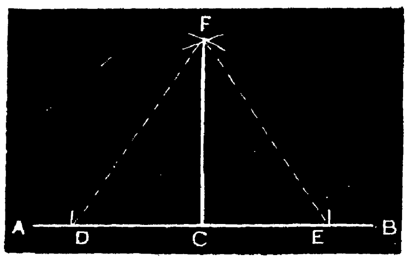


FIG. 117.

Part. Enun. Let AB be a given st. line of unlimited length and C a given pt. in it.

It is reqd. to draw a $\perp$ *to* AB *from* C.

Const. With C as centre and any radius desc. a ⊙ cutting AB in D and E.

With D and E as centres desc. equal ⊙s such that they intersect at a pt. F.

Join CF.

Then shall CF be the $\perp$ to AB from C.

Proof. Join FD, FE.

In $\triangle$s DCF, ECF

$\because \begin{cases} CD = CE & \text{. . . . . . . Const.} \\ CF \text{ is common.} & \\ FD = FE & \text{. . . . . . . Const.} \end{cases}$

$\therefore \triangle DCF \equiv \triangle ECF$ . . . . Th. 14.

so that $\llcorner$ DCF = $\llcorner$ ECF,

and these are adj. $\llcorner$s.

Hence CF is the $\perp$ to AB from C. Q.E.F.

Alternative Construction.

(When C is at or near one end of AB.)

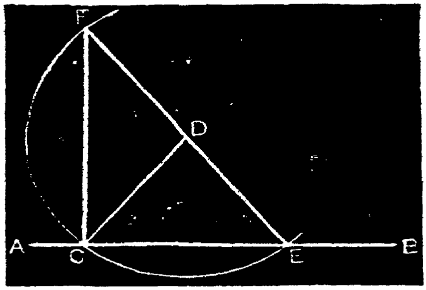


FIG. 118.

Const. Take any convenient pt. D outside AB.
Join CD.
With D as centre and radius = DC desc. a ⊙ cutting AB at E.
Join ED.
Produce ED to cut the ⊙ at F.
Join CF.
Then shall CF be the $\perp$ to AB from C.

Proof. $\because$ DC = DF . . . . Const.
$\therefore \angle$ DFC = $\angle$ DCF . . . . Th. 12.
And $\because$ DC = DE . . . . . . Const.
$\therefore \angle$ DEC = $\angle$ DCE . . . . Th. 12.
$\therefore \angle$ DFC + $\angle$ DEC = $\angle$ DCF + $\angle$ DCE
= $\angle$ FCE.
But $\angle$ DFC + $\angle$ DEC + $\angle$ FCE = 2 rt. $\angle$s Th. 8.
$\therefore \angle$ FCE = 1 rt. $\angle$,
that is, CF is the $\perp$ to AB from C. Q.E.F.

Exercises.

489. Draw a st. line at rt. $\angle$s to a given finite st. line from one of its extremities without producing the given st. line. (U. P. Eur. Schls. High.)

490. Draw an angle of 45° with ruler and compasses.

491. Draw an angle of $22\frac{1}{2}$° with ruler and compasses.

492. Draw a rt. $\angle$ and divide it into 2 parts so that one may be a seventh of the other.

*493. Construct a square having given a diag.

*494. Construct a rhombus having given its 2 diags.

Problem 4.

(Euc. I. 12.)

Gen. Enun. *To draw a straight line perpendicular to a given straight line of unlimited length from a given point outside it.*

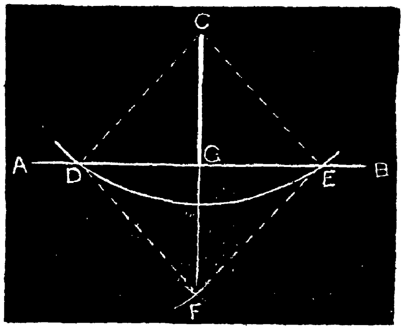


Fig. 119.

Part. Enun. Let AB be a given st. line of unlimited length and C a given pt. outside it.
It is reqd. to draw a $\perp$ *to* AB *from* C.

Const. With C as centre desc. a $\odot$ of such a radius that it cut AB in D and E.
With D and E as centres desc. equal $\odot$s such that they intersect at a pt. F.
Join CF cutting AB in G.
Then shall CG be the $\perp$ to AB from C.

Proof. Join CD, CE, FD, FE.
In $\triangle$s DCF, ECF,
$\because \begin{cases} DC = EC \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{Const.} \\ CF \text{ is common.} \\ DF = EF \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{Const.} \end{cases}$
$\therefore \triangle DCF \equiv \triangle ECF$ . . . . Th. 14.
so that $\angle DCF = \angle ECF$.

Again. In $\triangle$s DCG, ECG,
$\because \begin{cases} DC = EC \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{Const.} \\ CG \text{ is common.} \\ \text{contained } \angle DCG = \text{contained } \angle ECG \\ \qquad\qquad \text{Proved above.} \end{cases}$

∴ △ DCG ≡ △ ECG, . . . Th. 10.
so that ∟ CGD = ∟ CGE.
And these are adj. ∟s.
Hence CG is the ⊥ to AB from C. Q.E.F.

Alternative Construction.

(When C is opposite, or nearly opposite, one end of AB.)

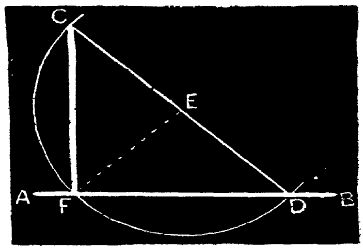


FIG. 120.

Const. Take any convenient pt. D in AB.
Join CD.
Bisect CD in E . . . . . . Prob. 2.
With E as centre and radius = EC desc. a ⊙ cutting AB in F.
Join CF.
Then shall CF be the ⊥ to AB from C.
Join FE and prove as in Prob. 3, alternative construction.

Exercises.

495. Find the orthocentre of a given △.

496. Through a given pt. draw a st. line equally inclined to 2 given intersecting st. lines. (Beng. Eur. Schls. Mid.)

* 497. From 2 given pts. on the same side of a given st. line, draw 2 st. lines meeting in it, but not in the same st. line, and making = ∟s with it. (Allah. Mat.)

* 498. From 2 given pts. on opp. sides of a given st. line, draw 2 st. lines meeting in it, but not in the same st. line, and making = ∟s with it.

* 499. From a given pt. to 2 || st. lines. draw 2 = st. lines at rt. ∟s to one another. (Allah. Mat.)

Problem 5.

(Euc. I. 23.)

Gen. Enun. *At a given point in a given straight line, to make an angle equal to a given angle.*

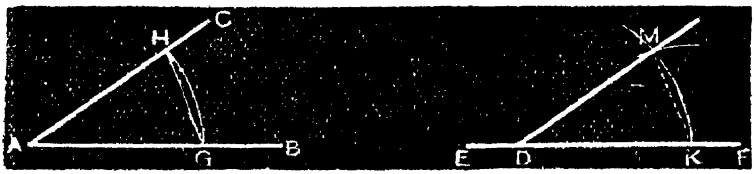


Fig. 121.

Part. Enun. Let D be a given pt. in the given st. line EF and BAC a given ∠.

It is reqd. to draw from D *a st. line making with* EF *an* ∠ = ∠ BAC.

Const. With A as centre, and any radius, desc. a ⊙ cutting AB, AC at G, H respy.

With D as centre, and the same radius, desc. a ⊙ cutting EF at K.

Join GH.

With K as centre and with radius = GH desc. a ⊙ cutting the ⊙ with centre D at M.

Join DM.

Then shall ∠ KDM = ∠ BAC.

Proof. Join KM.

In △s KDM, GAH,

∵ { DK = AG . . . . Const.
DM = AH . . . . Const.
KM = GH . . . . Const. }

∴ △ KDM ≡ △ GAH . . . Th. 14.

so that ∠ KDM = ∠ BAC. Q.E.F.

Exercises.

500. ABCD is a quadl. Construct another quad. ≡ ABCD.

501. Find a pt. in an arm of an ∠ equidistant from the vertex and from a given pt. in the other arm.

502. In the base BC, produced if necessary, of a $\triangle$ ABC, find a pt. D such that it is equidistant from A and C.

503. A is a given pt., and B is a given pt. in a given st. line; draw from A to the given st. line a st. line AP such that the sum of AP and PB = a given length > AB. (Bomb. Matric.)

504. A is a given pt., and B is a given pt. in a given st. line; draw from A to the given st. line a st. line AP such that the diffce. of AP and PB = a given length.

* 505. AB is a given st. line, and P is a given pt. From P draw 2 st. lines making $\angle$s with AB whose diffce. = a given $\angle$.

* 506. AB is a given st. line, and P is a given pt. From P draw 2 st. lines making $\angle$s with AB whose sum = a given $\angle$.

* 507. Divide a rt. $\angle$d $\triangle$ into 2 isos. $\triangle$s. (Bom. Previous.)

Problem 6.

(Euc. I. 31.)

Gen. Enun. *Through a given point to draw a straight line parallel to a given straight line.*

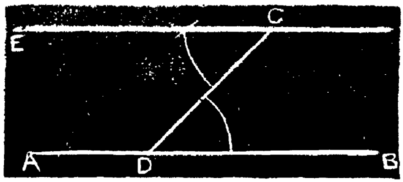


Fig. 122.

Part. Enun. Let C be a given pt. and AB a given st. line.
It is reqd. to draw through C *a st. line* $\parallel$ *to* AB.

Const. Take any pt. D in AB.
Join CD.
At the pt. C in the st. line CD make alt.
$\angle$ DCE = alt. $\angle$ CDB . . . . Prob. 5.
Then shall CE be $\parallel$ to AB.

Proof. $\because$ alt. $\angle$ DCE = alt. $\angle$ CDB . . . Const.
$\therefore$ CE is $\parallel$ to AB . . . . . Th. 4.

Q.E.F.

Exercises.

508. Construct a $\square$gm having given 2 adj. sides and their contained $\angle$.

509. Construct a rect. having given its base and height.

510. Construct a rhombus having given a side and an $\angle$.

511. Construct a square having given a side. (Euc. I. 46.)

512. From a given pt. draw a st. line to a given st. line making with it an $\angle$ = a given $\angle$.

513. AB is the hypot. of a rt. $\angle$d $\triangle$ ABC; find a pt. D in AB such that DB = $\perp$ from D on AC. (Calc. F. E.)

514. From a given pt. draw a st. line to two given $\parallel$ st. lines so that the part intercepted may be of a given length.

*515. Through a given pt. within a given $\angle$ draw a st. line bounded by the arms of the $\angle$ and bisected at the pt.

* 516. AB, AC are the = sides of an isos. △. Find pts. D, E in AB, AC respy. such that BD = DE = EC.

* 517. Find pts. P, Q in the arms AB, AC of an ∠ BAC such that PQ may be of a given length and ∥ to a given st. line.

* 518. Through a given pt. A draw a st. line to meet the arms of a given ∠ in B and C so that AB = BC.

* 519. In a △ ABC draw a st. line ∥ to AC and meeting AB, BC in D, E respy., so that DE = AD + CE. (Bom. Matric.)

* 520. In a △ ABC draw a st. line ∥ to AC and meeting AB, BC in D, E respy., so that DE = diffce. between AD and CE. (Mad. Matric.)

* 521. Inscribe a rhombus in a given △ so that one of the ∠s of the rhombus shall coincide with one of the ∠s of the △. (Mad. Matric.)

* 522. Draw a st. line equally inclined to 2 given intersecting st. lines so that the part intercepted between these lines may be of given length. (Mad. Matric.)

* 523. Draw a st. line DE ∥ to a side BC of a △ ABC and cutting the sides AB, AC in D and E so that BD + CE = a given length. (Mad. F. A.)

Problem 7.

Gen. Enun. *To divide a given finite straight line into any number of equal parts.*

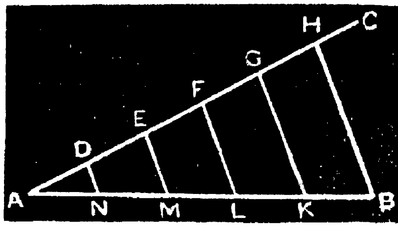


Fig. 123.

Part. Enun. Let AB be a given finite st. line.
It is reqd. to divide it into any number (say five) of equal parts.

Const. Through A draw AC making any $\angle$ with AB.
From AC cut off five equal parts of any length, AD, DE, EF, FG, GH.
Join HB.
Through D, E, F, G draw DN, EM, FL, GK ||s to HB and meeting AB in N, M, L, K respy. . Prob. 6.
Then shall AN = NM = ML = LK = KB.

Proof. $\because$ DN, EM, FL, GK, HB are ||s . . . Const.
And AD = DE = EF = FG = GH . . . Const.
$\therefore$ AN = NM = ML = LK = KB. . . Th. 23.

Q.E.F.

Exercises.

524. Trisect a given finite st. line.

525. Draw a finite line of any length and divide it into 7 = parts. (Punj. Mat.)

526. Draw a line = $\frac{3}{5}$ of a given finite line.

527. Divide a given finite line into 2 parts such that one part is $\frac{3}{4}$ of the other.

* 528. A, B, C are 3 pts. not in the same st. line. Show how to draw through A a st. line such that the $\perp$ on it from B shall be double the $\perp$ on it from C. (Mad. F. A.)

TRIANGLES.

PROBLEM 8.

Gen. Enun. *To construct a triangle having given two of its sides and the contained angle.*

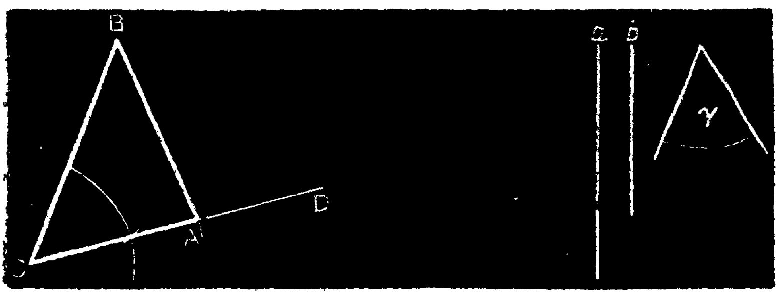


FIG. 124.

Part. Enun. Let a, b be 2 given st. lines and γ a given ∟.
It is reqd. to construct a △ having 2 sides = a, b respy. and the contained ∟ = ∟ γ.

Const. Draw any st. line CB = a.
At the pt. C in the st. line CB make ∟ BCD = ∟ γ . . . . Prob. 5.
From CD cut off CA = b.
Join BA.
Then shall △ ABC have the parts that are given.

Proof. ∵ { CB = a . . . . Const.
CA = b . . . . Const.
∟ BCA = ∟ γ . . . . Const. }
∴ △ ABC has the parts that are given. Q.E.F.

PROBLEM 9.

Gen. Enun. *To construct a triangle having given two of its angles and one side.*

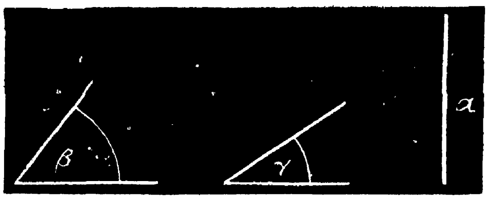


FIG. 125.

Part. Enun. Let β, γ be 2 given ∟s together < 2 rt. ∟s and a a given st. line.
It is reqd. to construct a △ *having* 2 ∟*s* = ∟ β, ∟ γ *respy. and a side* = a.

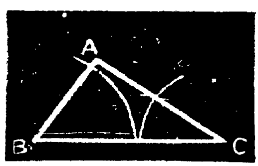


FIG. 126.

Const. (i) Draw any st. line BC = a. (Fig. 126.)
At the pt. B in the st. line BC make ∟ CBA = ∟ β . . . . Prob. 5.
At the pt. C in the st. line BC and on the same side of it as ∟ CBA make ∟ BCA = ∟ γ . . . . Prob. 5.

Then shall $\triangle$ ABC have the parts that are given.

Proof. $\because$ $\begin{cases} BC = a & \text{Const.} \\ \angle CBA = \angle \beta & \text{Const.} \\ \angle BCA = \angle \gamma & \text{Const.} \end{cases}$

$\therefore$ $\triangle$ ABC has the parts that are given.

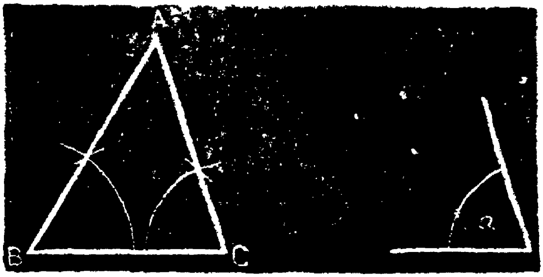


FIG. 127.

Const. (ii) Draw any st. line BC = a. (Fig. 127.)

At the pt. B in the st. line BC make $\angle$ CBA = $\angle \beta$ . . . . **Prob. 5.**

Let α be the $\angle$ such that $\angle \alpha + \angle \beta + \angle \gamma$ = 2 rt. $\angle$s.

At the pt. C in the st. line BC and on the same side of it as $\angle$ CBA make $\angle$ BCA = $\angle \alpha$ . . . . Prob. 5.

Then shall $\triangle$ ABC have the parts that are given

Proof. $\because$ $\angle$ BCA + $\angle$ CBA + $\angle$ BAC = 2 rt. $\angle$s . Th. 8.

$\therefore$ $\angle \alpha + \angle \beta + \angle$ BAC = 2 rt. $\angle$s.

But $\angle \alpha + \angle \beta + \angle \gamma$ = 2 rt. $\angle$s.

$\therefore$ $\angle$ BAC = $\angle \gamma$.

Hence in $\triangle$ ABC

$$\begin{cases} BC = a \\ \angle CBA = \angle \beta \\ \angle BAC = \angle \gamma \end{cases}$$

$\therefore$ $\triangle$ ABC has the parts that are given. Q.E.F.

PROBLEM 10.

(EUC. I 22.)

Gen Enun. *To construct a triangle having given its three sides.*

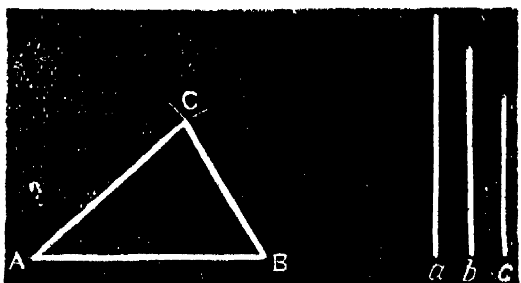


FIG. 128.

Part. Enun. Let a, b, c be 3 given st. lines of which any 2 are together > the 3rd.

It is reqd. to construct a △ having its sides = a, b, c respy.

Const. Draw any st. line AB = a.

With A as centre and radius = b desc. a ⊙.

With B as centre and radius = c desc. a ⊙.

Let these 2 ⊙s cut at C.

Join AC, BC.

Then shall △ ABC have the parts that are given.

Proof. ∵ $\begin{cases} AB = a & \text{Const.} \\ AC = b & \text{Const.} \\ BC = c & \text{Const.} \end{cases}$

∴ △ ABC has the parts that are given. Q.E.F.

Problem II.

(The Ambiguous Case.)

Gen. Enun. *To construct a triangle having given two of its sides and the angle opposite one of them.*

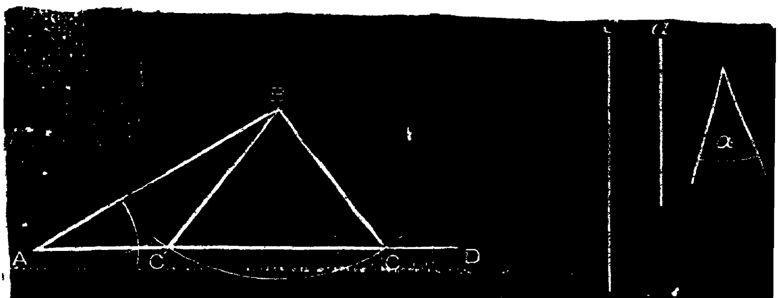


Fig. 129.

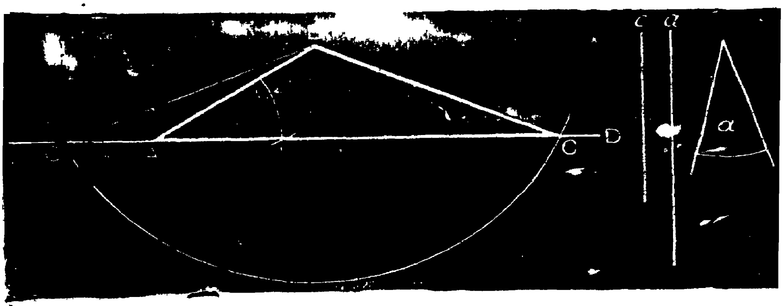
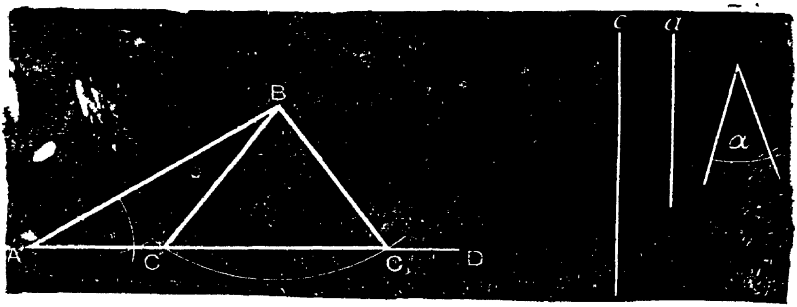


Fig. 130.

Part. Enun. Let a, c be 2 given st. lines and α a given ∠.
It is reqd. to construct a △ having 2 sides = a, c respy. and the ∠ opp. one of them (say a) = ∠ α.



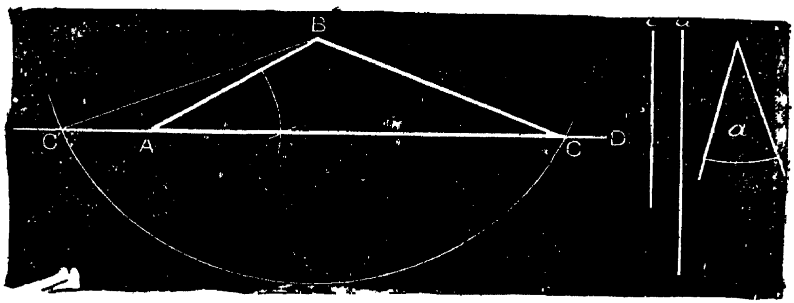


FIG. 130.

Const. Draw any st. line AB = c.

At the pt. A in the st. line AB make $\angle$ BAD = $\angle a$ . . . . Prob. 5.

With B as centre and radius = a desc. a $\odot$ cutting, if possible, DA, or DA produced, in 2 pts. C, C′.

Join BC, BC′.

Case (i) Let C, C′ be both on the same side of A as in Fig. 129.

Then shall $\triangle$ ABC and $\triangle$ ABC′ have the parts that are given.

Case (ii) Let C, C′ be on opp. sides of A as in Fig. 130.

Then shall $\triangle$ ABC have the parts that are given.

Proof. $\because$ in both Fig. 129 and Fig. 130

$$\left\{\begin{array}{ll} AB = c \quad . \quad . \quad . \quad . \quad . \quad . & \text{Const.} \\ BC = a \quad . \quad . \quad . \quad . \quad . \quad . & \text{Const.} \\ \angle BAC = \angle a \quad . \quad . \quad . \quad . \quad . & \text{Const.} \end{array}\right.$$

$\therefore$ $\triangle$ ABC has the parts that are given.

Similarly it may be shown that $\triangle$ ABC′ in Fig. 129 has the parts that are given Q.E.F.

Problem 12.

Gen. Enun. *To construct a right-angled triangle having given its hypotenuse and one of its sides.*

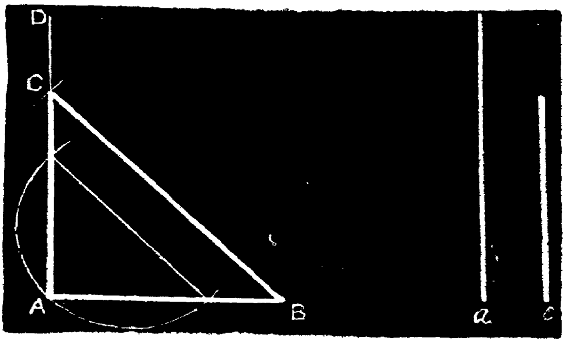


Fig. 131.

Part. Enun. Let a, c be 2 given st. lines of which $a > c$.
It is reqd. to construct a rt. ∟d △ having its hypot. = a and a side = c.

Const. Draw any st. line AB = c.
At the pt. A in the st. line AB draw AD ⊥ to AB . . . . Prob. 3.
With B as centre and radius = a desc. a ⊙ cutting AD in C.
Join BC.
Then shall △ ABC be a rt. ∟d △ having the parts that are given.

Proof. ∵ ∟ BAC is a rt. ∟ . . . . . . Const.
AB = c . . . . Const.
BC = a . . . . Const.
∴ ABC is a rt. ∟d △ having the parts that are given.

Q.E.F.

Exercises.

529. Construct a △ equal in all respects to a given △.
530. Construct an equilat. △ having given its base. (Euc. I. 1.)
531. Construct an isos. △ having given its base and a side.

532. Construct an isos. $\triangle$ on a given base and having each side double its base. (U. P. Eur. Schls. Mid.)

533. Construct an isos. $\triangle$ having given its base and altitude.

534. Construct an isos. $\triangle$ having given its base and perimeter.

535. Construct a rt. $\angle$d $\triangle$ having given its hypot. and an acute $\angle$.

536. Construct a rt. $\angle$d $\triangle$ having given its hypot. and base.

537. Construct an isos. rt. $\angle$d $\triangle$ having given its hypot.

538. Construct an equilat. $\triangle$ having given its altitude.

539. Construct a rhombus having given an $\angle$ and its perimeter.

540. Construct an equilat. $\triangle$ whose perimeter is equal to that of a given $\triangle$.

541. Construct a $\triangle$ having given its 2 sides and altitude.

542. Construct a $\triangle$ ABC having given AB, BC and the median through A.

543. Construct a $\triangle$ having given its base, altitude and a side.

544. Construct a ▱gm having given one side and the 2 diags. (Bomb. Matric.)

545. Construct a rt. $\angle$d $\triangle$ having given its hypot. and having one acute $\angle$ = $\frac{1}{2}$ the other.

546. Construct a rt. $\angle$d $\triangle$ having given its base and the $\perp$ from the rt. $\angle$ on the hypot. In which case will the construction fail? (Punj. Mat.)

547. Construct a rt. $\angle$d $\triangle$ having given its base and the $\angle$ opp. the base.

548. Construct a rhombus having each of its sides = one of its diags.

549. Construct an isos. $\triangle$ having given the vert. $\angle$ and altitude.

550. Construct an isos. $\triangle$ whose vert. $\angle$ is 4 times each of its base $\angle$s.

551. Construct an isos. $\triangle$ having given its base and vert. $\angle$.

552. Construct an isos. $\triangle$ having given its base and the sum of its vert $\angle$ and a base $\angle$.

* 553. Construct an isos. $\triangle$ having given its base and the sum of its altitude and one of its sides.

* 554. Construct an isos. $\triangle$ having given its altitude and perimeter.

* 555. Trisect a rt. $\angle$. (Calc. Mat.)

* 556. Construct a $\triangle$ having given its altitude and the $\angle$s at the base.

* 557. Construct a $\triangle$ having given its base, one of the $\angle$s at the base and the sum of the sides. (Allah. Mat.)

* 558. Construct an obtuse $\angle$d $\triangle$ having given its greatest side, an adj. $\angle$ and the diffce. of the remg. sides.

* 559. Construct a $\triangle$ having given its perimeter and base $\angle$s.

* 560. Construct a rt. $\angle$d $\triangle$ having given its hypot. and the sum of its base and perpendicular.

* 561. Construct a $\triangle$ having given its base, the diffce. of its base $\angle$s and the diffce. of its remaining sides. (Mad. Matric.)

*562. Construct a rt. ∟d △ having given the perimeter and one acute ∟. (Mad. Matric.)

*563. Construct an equilat. △ having given a pt. on each of its sides and the line along which its base lies.

*564. Describe a regular hexagon without drawing a ⊙. (Punj. Eur. Schls. High.)

*565. Construct a △ having given its medians. (Bom. Previous.)

*566. On a given st. line as diag. desc. a rhombus with ∟s 60°, 60°, 120°, 120°. (U. P. Eur. Schls. Mid.)

*567. Construct a square having given the diffce. between a diag. and one side.

*568. Trisect any angle using compasses and *graduated* ruler.

QUESTIONS FOR EXAMINATION.—XXX.

1. Distinguish between a problem and a theorem.

2. What is the *datum* and what is the *quæsitum* in the enunciation of Problem 1?

3. What simple constructions are assumed as possible in the solution of problems, and what instruments are available?

4. Define the term "postulate".

5. The equal circles described with centres D and E, in the construction of Problem 1, intersect in *two* points. Prove that both points lie on the bisector of the angle.

6. Show, by a figure, the necessity of describing equal circles with centres D and E, in the construction of Problem 1, "such that they intersect".

7. Prove that AF in figure 115 is part of the locus of points equidistant from AB and AC. Where is the other part?

8. Prove that FA produced, in figure 115, will bisect the reflex angle BAC.

9. Draw an angle on paper, cut it out and bisect it by folding. Prove your result.

10. Prove the construction of Problem 1 by folding about the supposed bisector.

11. Prove that CD in figure 116 is the *perpendicular* bisector of AB.

12. The radii of the equal circles described with A and B as centres in the construction of Problem 2 must be greater than half AB. Give a reason for this.

13. Prove that CD in figure 116 is the locus of points equidistant from A and B.

14. Draw a straight line on tracing paper and bisect it by folding. Prove your result.

15. Prove the construction of Problem 2 by folding about the supposed bisector.

16. Deduce Problem 3 as a special case of Problem 1.

17. What is a line "of unlimited length"? (U. P. Eur. Schls. Mid.)

18. How long at least must the radii of the equal circles be that are described in the construction of Problem 8?

19. Why is the line AB in Problem 4 given as of unlimited length? (U. P. Eur. Schls. Mid.)

20. Show, by a figure, that the construction of Problem 4 fails if the circle described with centre C does not cut AB.

21. Define equal circles. What equal circles are described in the construction of Problem 5?

22. Enunciate the theorem on which the proof of Problem 6 depends.

23. Of what points is EC in the figure of Problem 6 part of the locus? Where is the other part?

24. Enunciate the theorem on which the proof of Problem 7 depends.

25. Two triangles are constructed from the same data as the triangle in the figure of Problem 8. By what theorem do we know that they are congruent?

26. What is the reason for the condition that the two given angles in Problem 9 are together less than two right angles?

27. Enunciate the theorem on which the construction of Problem 9, Case (ii), depends.

28. What is the reason for the condition that the sum of any two of the given straight lines in Problem 10 must be greater than the third? (Beng. Eur. Schls. Mid.)

29. Prove, by means of a figure, that three lines might be taken such that it would be impossible to construct a triangle with sides equal to them. (Beng. Eur. Schls. Mid.)

30. When is a problem said to be *indeterminate*? Give an example.

31. If, in the figures of Problem 11, a is less than the perpendicular from B on AD, prove that the construction fails.

32. Wherein lies the ambiguity of Problem 11?

33. If, in the figures of Problem 11 (i) $a = c$, (ii) $a =$ the perpendicular from B on AD, prove that the ambiguity disappears.

34. In what cases can a triangle be constructed, having given three of its six parts?

35. In what cases can a triangle *not* be constructed, having given three of its six parts?

36. State the smallest number of measurements necessary to determine a quadrilateral fully, and show that they are sufficient. Examine if three angles and one side would do. (Beng. Eur. Schls. Mid.)

PART II.

SCALES.

A SCALE showing only primary and secondary divisions is called a plain scale. Your graduated ruler, for example, is a plain scale, because it only shows inches and tenths of an inch, centimetres and tenths of a centimetre.

Diagonal scales show primary, secondary and tertiary divisions, and are used when very accurate measurement is required. Fig. 132 is a diagonal scale showing inches, tenths of an inch and hundredths of an inch, and consists of:—

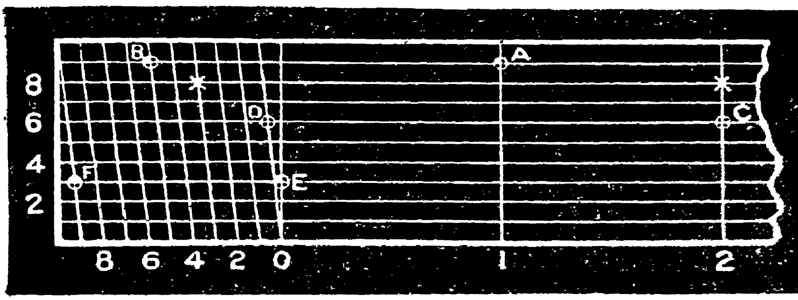


FIG. 132.

(1) A bottom line divided as a plain scale to show inches and tenths of an inch, but only the first primary division from the left is divided into secondary divisions.
(2) Verticals through the primary divisions of the bottom line.
(3) Parallels to the bottom line at convenient equal distances. In this case there are ten parallels because there are ten hundredths of an inch in one-tenth of an inch.
(4) The top parallel divided in the same way as the bottom line.
(5) Diagonals through the secondary divisions of the bottom line

The principle of the diagonal scale depends upon the theorem :—

The straight line joining the middle points of the sides of a triangle is equal to half the base (Ex. 414),

from which it follows in Fig. 133, which is part of Fig. 132 enlarged, that

$$Aa = \tfrac{1}{2}\,Bb = \tfrac{1}{3}\,Cc = \tfrac{1}{4}\,Dd = \tfrac{1}{5}\,Ee = \tfrac{1}{6}\,Ff = \tfrac{1}{7}\,Gg = \tfrac{1}{8}\,Hh = \tfrac{1}{9}\,Kk = \tfrac{1}{10}\,Ll.$$

(The proof of this is left as an exercise for the student.)

But L*l* represents ·1 in.

$$\text{Therefore}\left\{\begin{array}{lll} Aa & \text{rep.} & \cdot01 \text{ in.} \\ Bb & ,, & \cdot02\ ,, \\ Cc & ,, & \cdot03\ ,, \\ Dd & ,, & \cdot04\ ,, \\ Ee & ,, & \cdot05\ ,, \\ Ff & ,, & \cdot06\ ,, \\ Gg & ,, & \cdot07\ ,, \\ Hh & ,, & \cdot08\ ,, \\ Kk & ,, & \cdot09\ ,, \end{array}\right.$$

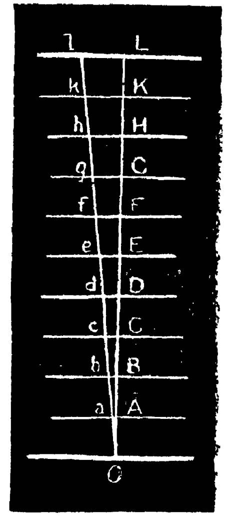


FIG. 133.

Exercises.

569. Set off a distance of 2·38 in. from the diagonal scale (Fig. 132) with your dividers.

Set off 2·3 in. from the bottom line as with a plain scale, move the right-hand point up vertical 2 until it reaches parallel 8, and then extend the dividers until the left hand point reaches the intersection of parallel 8 and diagonal 3. The required distance is shown in Fig. 132 thus, * . . . . . . *.

570. Take the following lengths from the diagonal scale (Fig. 132) and draw lines of these lengths :—

(i) 1·24 in.
(ii) 2·67 in.
(iii) 2·18 in.
(iv) 1·07 in.
(v) 2·90 in.
(vi) 0·77 in.

571. Read off the distances between A and B, C and D, E and F marked on the diagonal scale (Fig. 132).

572. Find the lengths of the lines I, II, III (Fig. 134) by applying them to the diagonal scale (Fig. 132).

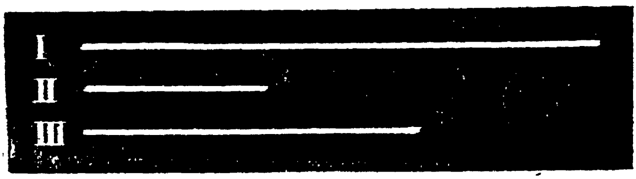


FIG. 134.

573. Take the following lengths from your graduated ruler, and find their equivalents in inches to two places of decimals by applying them to the diagonal scale (Fig. 132):—

(i) 6 cm.
(ii) 3·5 cm.
(iii) 43 mm.

DRAWING TO SCALE.

If a drawing of a large object is made so that any line an inch long taken from the drawing corresponds to and represents a line a mile long in the object, the drawing is said to be "drawn to scale," and, since every line in the drawing is $\frac{1 \text{ in.}}{1 \text{ mile}} = \frac{1 \text{ in.}}{1760 \times 36 \text{ in.}}$ $= \frac{1}{63360}$ of its actual length in the object, this fraction is called the representative fraction (R. F.) of the scale. Thus in a map drawn to the scale of 1 inch to 50 miles, we have

$$\text{R. F.} = \frac{1 \text{ in.}}{50 \text{ miles}} = \frac{1}{50 \times 1760 \times 36} = \frac{1}{3168000},$$

and in a map drawn to the scale of 1·25 inches to 7 furlongs, we have

$$\text{R. F.} = \frac{1{\cdot}25 \text{ in.}}{7 \text{ fur.}} = \frac{1{\cdot}25}{7 \times 220 \times 36} = \frac{1}{44352}.$$

N.B.—The numerator of the R. F. is always unity.

Exercises.

574. Drawings are made to the following scales. Work out the R. F. in each case:—

(i) 1 in. = 1 yd.
(ii) 1 in. = 1 mile.
(iii) 1 in. = 8·5 ft.
(iv) $\frac{1}{8}$ in. = $1\frac{1}{2}$ furlongs.
(v) 3·5 in. = 15 poles.
(vi) 2·3 in. = 1 metre (1 metre = 39·37 in.).

575. The R. F. of a map is $\frac{1}{360{,}000}$. What length on the map will represent 21 miles?

576. The distance between two towns as shown on a map is 2·5 in. The actual distance between them is 94 miles. Find the R. F. of the map.

577. The R. F. of a French map is $\frac{1}{700{,}000}$. How many English miles apart are two towns which are found on the map to be distant 2·3 cm. from each other (1 metre = 39·37 in.)?

578. A square has a side of 14 in. Construct it to the scale of 1 in. to 1 ft., and measure the diag. (Camb. Loc. Prel.)

579. A square fort of 100 yds. side is to be defended by muskets of 200 yds. range. Show by a diagram how much ground will be under range. Scale 100 yds. to 1 in. (Coll. of Precep. Jun.)

580. **A, B, C, D, E, F** are the corners of a regular hexagon each side of which is 11 miles. Construct the hexagon to the scale of 1 in. to 10 miles, and measure the lengths of **AC** and **AD** to the nearest mile. (Camb. Loc. Prel.)

581. A yacht sails successively 2 miles N.W., $3\frac{1}{2}$ miles S.W., 1 mile due S., $2\frac{1}{4}$ miles due E. Draw a plan of the course to scale 1 in. = 1 mile, and find by measurement the distance between the yacht's first and last position.

582. A ladder reaches from a spot 24 ft. distant from the base of a house to a window 30 ft. from the ground. Find by drawing and measurement the length of the ladder. Scale 12 ft. to 1 in. (Coll. of Precep. Sen.)

583. Six telegraph posts, each 25 ft. high, are to be placed at equal distances apart along a straight, level road. The first and sixth are 350 yds. from each other Make a drawing to show the positions of the posts to a scale of 80 yds. to 1 in. (Coll. of Precep. Sen.)

584. Find the representative fraction of the accompanying scale.

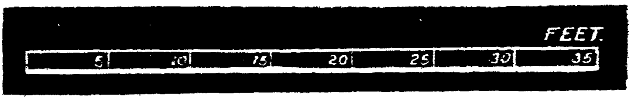


FIG. 135.

585. Sketch the floor of the room in which you are sitting and make a plan of it. Scale 20 ft. = 1 in.

CONSTRUCTION OF SCALES.

Having decided upon what scale a map or plan is to be drawn, the scale itself must be constructed. The following examples illustrate the methods of construction both for plain scales and for diagonal scales :—

Example 1. Construct a scale of 100 ft. to an inch to shew a length of 10 ft. Let the scale itself be 3 in. long.

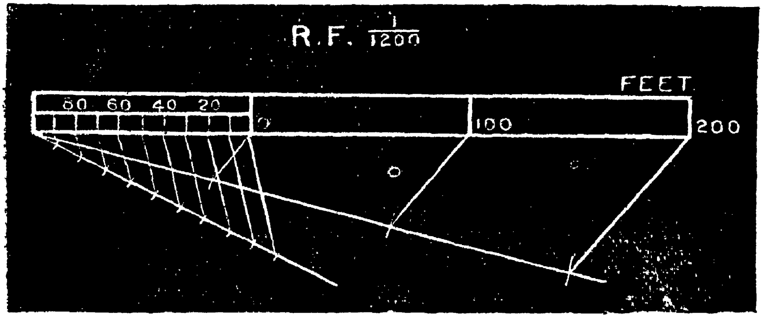


FIG. 136.

(1) Find the representative fraction

$$R.\ F. = \frac{1\text{ in.}}{100\text{ ft.}} = \frac{1\text{ in.}}{100 \times 12\text{ in.}} = \frac{1}{1200}.$$

(2) Find the distance represented by the whole length of the scale.

1 in. represents 100 ft.

∴ 3 in. ,, 300 ft.

(3) Draw a line 3 in. long and divide it into primary divisions of convenient lengths—in this case into three primary divisions, so that each division represents 100 ft.

(4) Divide the first primary division from the left into ten secondary divisions so that each secondary division represents 10 ft.

(5) Number the divisions and complete the scale as in Fig. 136.

Example 2. The representative fraction of a plan is $\frac{1}{125}$. Construct a scale to show feet. Let the scale itself be about 3 in. long.

(1) Determine the whole length of the scale and the distance represented by it.

$$R.\ F. = \tfrac{1}{125}.$$

∴ 1 in. represents 125 in.

∴ 3 in. ,, 375 in. = $31\frac{1}{4}$ feet

Now $31\frac{1}{4}$ ft. is an awkward distance to be represented by the whole length of the scale, so we assume the nearest

convenient number and then determine by proportion what the whole length of the scale will become.

Assume 30 ft.

$$31\tfrac{1}{4}\text{ ft.} : 30\text{ ft.} = 3\text{ in.} : \frac{30 \times 3}{31\tfrac{1}{4}}\text{ in.}$$

$$\text{and } \frac{30 \times 3}{31\tfrac{1}{4}}\text{ in.} = 2{\cdot}88\text{ in.}$$

Thus the whole length of the scale will be 2·88 in. and the distance represented by it will be 30 ft.

(2) Draw a line 2·88 in. long and divide it into primary divisions of convenient lengths—in this case into three

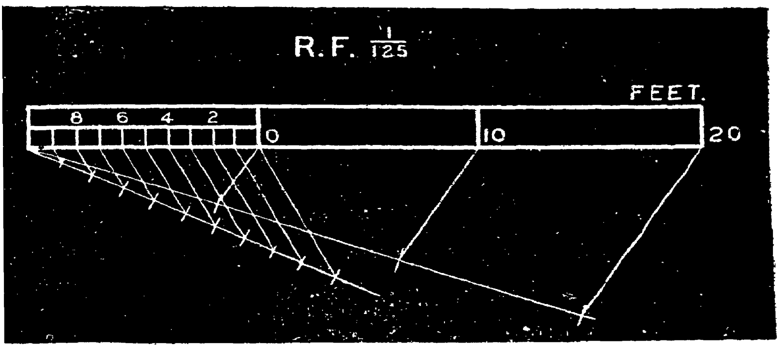


Fig. 137.

primary divisions, so that each division represents 10 ft.

(3) Divide the first primary division from the left into ten secondary divisions so that each secondary division represents 1 ft.

(4) Number the divisions and complete the scale as in Fig. 137.

Example 3. Construct a diagonal scale of 1 mile to an inch to show miles, tenths of a mile and hundredths of a mile. Let the scale itself be 3 in. long.

(1) Draw a bottom line 3 in. long and divide it as a plain scale R. F. $\frac{1\text{ in.}}{1\text{ mile}} = \frac{1}{63360}$ to show miles and tenths of a mile.

(2) Draw verticals through the primary divisions.

(3) Draw ten parallels to the bottom line at convenient equal intervals taken on the first vertical on the left.

(4) Join the first secondary division from the left to the ex-

treme left of the top parallel and draw diagonals parallel to this from the remaining secondary divisions.

(5) Number the divisions and complete the scale as in Fig. 138.

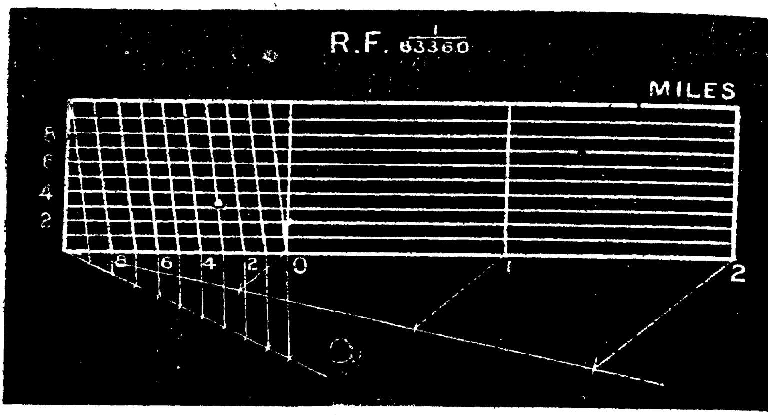


Fig. 138.

Example 4. Construct a diagonal scale R. F. $\frac{1}{95}$ to show yards, feet and inches. Let the scale itself be about 3 in. long.

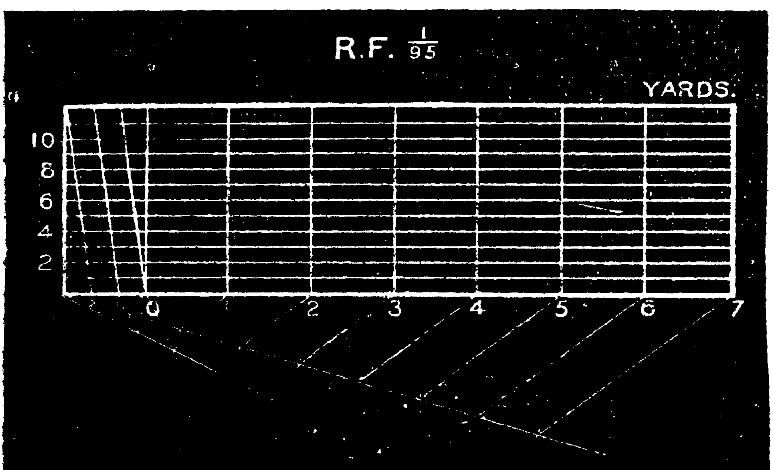


Fig. 139.

(1) Determine the whole length of the scale and the distance represented by it.

$$\text{R. F.} = \tfrac{1}{95}.$$

$\therefore$ 1 in. represents 95 in.

$\therefore$ 3 in. ,, 285. in. = $7\frac{11}{12}$ yds.

Now $7\frac{11}{12}$ yds. is an awkward distance to be represented by the whole length of the scale so we assume the nearest convenient number and then determine, by proportion, what the whole length of the scale will become.

Assume 8 yds.

$$7\tfrac{11}{12}\text{ yds.} : 8\text{ yds} = 3\text{ in.} : \frac{3 \times 8}{7\frac{11}{12}}\text{ in.}$$

and $\dfrac{3 \times 8}{7\frac{11}{12}}$ in. = 3·03 in. (Correct to the nearest hundredth of an inch.)

Thus the whole length of the scale will be 3·03 in. and the distance represented by it will be 8 yds.

(2) Draw a bottom line 3·03 in. long and divide it as a plain scale R. F. $\frac{1}{95}$ into primary and secondary divisions so as to represent yards and feet, then complete the scale and number the divisions as in Fig. 139.

Exercises.

586. Draw a st. line 4 in. long. This line represents 5 ft. Divide it to show feet and inches, and complete and figure the scale. (Camb Loc. Prel.)

587. Construct a plain scale R. F. $\frac{1}{840}$ to show 100 yds.

588. Construct a plain scale R. F. $\frac{1}{40}$ to measure yards and feet. The scale must be long enough to measure 5 yds. (Oxf. Loc. Prel.)

589. Draw a plain scale of $\frac{7}{8}$ in. to 1 ft. to show feet and inches, 5 ft being the greatest distance shown. (Camb. Loc. Prel.)

590. Construct a plain scale R. F. $\frac{1}{25}$ to show feet and inches.

591. Draw a plain scale of $1\frac{1}{4}$ in. to 1 ft. to show feet and inches 4 ft. being the greatest distance shown. (Camb. Loc. Prel.)

592. Make a plain scale R. F. $\frac{1}{10}$ to represent feet and inches. The scale must be long enough to measure 5 ft. (Oxf. Loc. Prel.)

593. Draw a scale of furlongs and chains to accompany a map on which 3 furlongs 6 chains measures $2\frac{1}{4}$ in. (Oxf. Loc. Sen.)

594. Draw a st. line $4\frac{1}{2}$ in. long to represent 5 miles. Construct on this line a plain scale, showing miles and furlongs, to measure distances up to 5 miles.

A man walks from A 2 miles in a northerly direction to B; he then walks 3 miles 2 furlongs in an easterly direction to C. By use of the scale, find to the nearest furlong the distance from A to C. (Camb. Loc. Jun.)

595. Draw a st. line of length $4\frac{1}{2}$ in. to represent 40 miles. Construct on this a plain scale, showing single miles, to measure up to 40 miles.

On a map two places B and C are found to be respy. 25 miles and 30 miles from A and $\angle$ BAC = 40°. By use of the scale, find the distance from B to C to the nearest mile. (Camb. Loc. Jun.)

596. Construct a plain scale R. F. $\frac{1}{25}$ to show decimetres (1 metre = 39·37 in.) and use it to draw a quadl. ABCD from the following data:—

AB = 1·9 metres AD = 1·4 metres $\angle$ ABC = 90° $\angle$ BCD = 70° $\angle$ CDA = 90°.

Find by measurement the remaining sides of the quadl.

597. Draw a st. line $4\frac{1}{2}$ in. long to represent 15 miles. Construct a diagonal scale upon this line to show miles and furlongs. (Camb. Loc. Jun.)

598. Construct a diagonal scale R. F. $\frac{1}{120}$ to show feet and inches, and from it take off a length of 36 ft. 7 in.

599. Construct a diagonal scale R. F. $\frac{1}{25}$ to show metres, decimetres and centimetres (1 metre = 39·37 in.). Scale to be about 6 in. long.

600. Construct a diagonal scale, 1 in. = 8 chains, to show 6 paces (1 pace = 30 in. and 1 chain = 22 yds.). Scale to be about 6 in. long.

601. Construct a diagonal scale R. F. $\frac{1}{110}$ to show poles, yards and feet. Scale to be about 6 in. long.

EXPERIMENTAL SECTION.

COORDINATES.

Exp. 123. On a piece of squared paper draw two straight lines X'OX and Y'OY at right angles to one another as in Fig. 140.

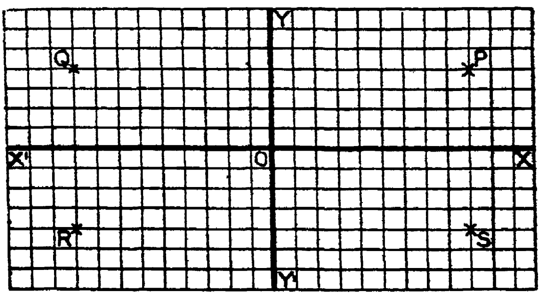


FIG. 140.

From O in Fig. 140 count *to the right* 9 divisions, then *up* 4 divisions, and so arrive at a point P. Mark P in your figure.

The numbers 9 *to the right* and 4 *up* determine the point P and are called its coordinates with respect to the coordinate axes X'OX and Y'OY (known respectively as the x-axis and y-axis), while O is called the origin of coordinates.

Exp. 124. From O in Fig. 140 count *to the left* 9 divisions, then *up* 4 divisions, and so arrive at a point Q, whose coordinates are 9 *to the left* and 4 *up*. Mark Q in your figure.

Exp. 125. From O in Fig. 140 count *to the left* 9 divisions, then *down* 4 divisions, and so arrive at a point R, whose coordinates are 9 *to the left* and 4 *down*. Mark R in your figure.

Exp. 126. From O in Fig. 140 count *to the right* 9 divisions, then *down* 4 divisions, and so arrive at a point S, whose coordinates are 9 *to the right* and 4 *down*. Mark S in your figure.

. Distances measured *to the right* or *up* are called positive and *to the left* or *down* negative, so that, for convenience, the coordinates of the point P in Fig. 140 are written (9, 4), of the point Q (−9, 4), of the point R (−9, −4) and of the point S (9, −4).

Coordinates *to the right* or *left* are called the abscissæ or x-coordinates of the points to distinguish them from coordinates *up* or *down*, which are called their ordinates or y-coordinates.

Notice that in describing a point its abscissa is always given first and its ordinate last. Thus the point whose abscissa is 8 and whose ordinate is 5 is written (8, 5), while the point whose position is unknown is written (x, y) where x denotes the abscissa and y the ordinate.

Exp. 127. Copy Fig. 141 on squared paper and write down the

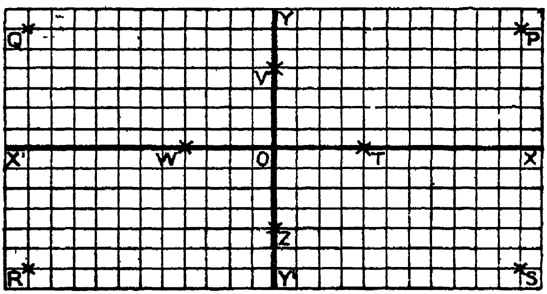


FIG. 141.

coordinates of the points P, Q, R, S, T, V, W, Z, O.

Exp. 128. Draw coordinate axes on a sheet of squared paper and plot the following points with reference to them :—

(i) (4, 10), (4, 5), (4, 0), (4, −5), (4, −10).
(ii) (10, 4), (5, 4), (0, 4), (−5, 4), (−10, 4).
(iii) (6, 6), (3, 3), (0, 0), (−3, −3), (−6, −6).
(iv) (3, 5·5), (1, 5), (−1, 4), (−3·5, 2), (−5, 0).

Exp. 129. Plot (7, 3) and (−7, −3) on squared paper and prove theoretically that they are equidistant from (0, 0).

Exp. 130. Plot (6, −5) and (−6, 5) on squared paper and prove theoretically that they are collinear with (0, 0).

Exp. 131. Plot (10, 0), (0, 3), (3, 0) and (0, 10) on squared paper and prove theoretically that the distance between the first two is equal to the distance between the last two.

Exp. 132. Plot (12, 9) on squared paper and prove theoretically that (8, 6) is a point of trisection of the line joining (12, 9) to (0, 0).

Exp. 133. Show by a diagram on plain paper that if a point lies on the x-axis its ordinate is 0, and if it lies on the y-axis its abscissa is 0.

Exp. 134. Show by a diagram on plain paper that the line joining (a, b) and $(-a, b)$ is bisected by the y-axis at right angles.

Exp. 135. Show by a diagram on plain paper that the line joining (a, b) and $(a, -b)$ is bisected by the x-axis at right angles.

Exp. 136. Show by a diagram on plain paper that the line joining $(0, 0)$ and (a, a) is bisected at right angles by the line joining $(0, a)$ and $(a, 0)$.

Exp. 137. Show by a diagram on plain paper that the line joining $(0, a)$ and $(b, 0)$ is equal to the line joining $(0, 0)$ and (b, a).

Exp. 138. Plot the following pairs of points on "inch" paper (that is squared paper ruled in inches and tenths of an inch), and express the distance between each pair in inches and decimals of an inch:—

(i) (3, 8) and (14, 8).
(ii) (−6, 4) and (−6, 12).
(iii) (−4, 9) and (−4, −6).
(iv) (−5, −3) and (−5, −12).
(v) (3·5, 6) and (9, 6).
(vi) (7, 6·9) and (7, −5·2).

Exp. 139. Plot and give the coordinates of the point where :—

(i) the line joining (5, 5) and (−7, −7) cuts the line joining (8, 3) and (−8, −3);
(ii) the line joining (6, 9) and (−4, 9) cuts the line joining (0, 0) and (0, 10);
(iii) the line joining (−3, 7) and (−3, −6) cuts the line joining (0, 0) and (9, 0);
(iv) the line joining (9, 4) and (2, 8) cuts the x-axis;
(v) the line joining (−11, 2) and (−3, 7) cuts the y-axis;
(vi) the line joining (9, 9) and (−9, −3) cuts the line joining (−3, 8) and (9, −2).

Note.—When using squared paper we shall be able, with practice to read off tenths but not hundredths of a division. On inch paper this means an approximation to two decimal places of an inch.

Exp. 140. Plot the points P (2, 4) and Q (10, 12). Bisect the

line joining P, Q. Let R be the mid-point of PQ. Draw the ordinates PM, QN, RS of P, Q, R respectively. Now prove *theoretically* that $RS = \frac{PM + QN}{2} = \frac{4 + 12}{2} = 8$ and verify *by*

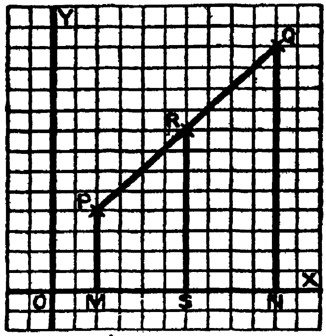


FIG. 142.

counting. Prove in a similar way that the abscissa of R is 6 and verify by counting.

Exp. 141. Plot the following pairs of points and find the coordinates of the points midway between each pair *by calculation* as in Exp. 140. Verify *by bisection and counting*:—

(i) (2, 9) and (8, 1).
(ii) (0, 0) and (10, 6).
(iii) (− 4, 3) and (15, 14).
(iv) (3, − 4) and (− 7, − 6).

Exp. 142. Show by a figure that (− 4, 3), (− 4, − 5), (8, 3), (8, − 5) are the vertices of a rectangle, and find by calculation the coordinates of the point of intersection of the diagonals. Verify by drawing and counting.

Exp. 143. Show by a figure that (7, 9), (7, 4), (7, 0), (7, − 5) lie on the locus of points equidistant from (1, 5) and (13, 5).

Exp. 144. Show by a figure that (− 1, − 2), (− 7, − 2), (− 7, − 8), (− 1, − 8) are equidistant from (− 4, − 5).

Exp. 145. Find by construction and counting the coordinates of the point equidistant from (2, 2) (11, 2), (11, 5).

Exp. 146. Show by a figure the locus of points distant 0·6 in. from (− 3, − 7) on inch paper.

Exp. 147. Find by construction and counting the coordinates of points distant 0·5 in. from (8, 2) and (3, 6) on inch paper.

Exp. 148. Find by construction and counting the coordinates of points distant 0·75 in. from (5, 3) and 0·6 in. from the x-axis on inch paper.

Exp. 149. Find by construction and counting the coordinates of the orthocentre of the triangle whose vertices are (2, 7), (– 9, – 3), (5, – 10).

Exp. 150. Find by construction and counting the coordinates of the centroid of the triangle whose vertices are (2, 2), (13, 7), (11, – 8).

GRAPHS.

ALGEBRAICAL EQUATIONS.

Exp. 151. Plot a series of points such that for each point

$$x\text{-coordinate} + y\text{-coordinate} = 9,$$

for example $(-3, 12)$, $(-2, 11)$, $(-1, 10)$, $(0, 9)$, $(1, 8)$, $(2, 7)$, $(3, 6)$.

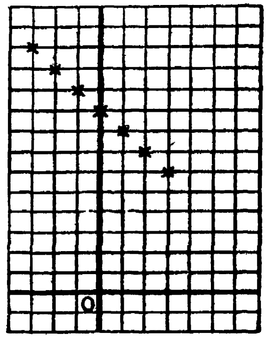


FIG. 143.

Notice that the locus of these points is a *straight line*. Since for every point on this line $x + y = 9$ where x and y stand for the x and y coordinates of the point, this line is called the graph of the equation $x + y = 9$.

Exp. 152. Plot a series of points such that for each point

$$x\text{-coordinate} = 6,$$

and the y-coordinate is anything whatever, for example $(6, -5)$, $(6, -1)$, $(6, 0)$, $(6, 2)$, $(6, 4)$, $(6, 8)$.

Notice that the locus of these points is a straight line parallel to the y-axis, and that the straight line is the graph of the equation $x = 6$, because this equation is satisfied by the coordinates of all points on the line.

Exp. 153. Plot a series of points such that for each point

$$y\text{-coordinate} = 2 \times x\text{-coordinate} + 3,$$

that is, plot the graph of the equation.

$$y = 2x + 3.$$

Corresponding values of x and y can be tabulated thus :—

Values of x	-3	-2	-1	0	1	2	3
Corresponding values of y .	-3	-1	1	3	5	7	9

It is now an easy matter to plot a series of points whose coordinates satisfy the equation and then join them by a continuous line.

Exp. 154. Plot the graph of the equation

$$x^2 + 2x - y = 3,$$

that is,

$$y = x^2 + 2x - 3$$

Corresponding values of x and y can be tabulated thus :—

Values of x	-3	-2	-1	0	1	2	3
Corresponding values of x^2 .	9	4	1	0	1	4	9
Corresponding values of $2x$.	-6	-4	-2	0	2	4	6
Corresponding values of y $(= x^2 + 2x - 3)$	0	-3	-4	-3	0	5	12

Exp. 155. Plot the graphs of the following equations :—

(i) $x + 3 = 0$.
(ii) $y = 11$.
(iii) $x = y$.
(iv) $x + y = 12$.
(v) $y - 3x + 4 = 0$
(vi) $y = x^2$.
(vii) $x = y^2 - 1$.
(viii) $x^2 + y^2 = 25$.

Exp. 156. Plot the graphs of the equations

$$\left.\begin{aligned} x - 4y + 5 &= 0 \\ x + y - 5 &= 0 \end{aligned}\right\},$$

and hence find, by counting, the coordinates of their point of intersection. Verify by solving algebraically.

Exp. 157. Solve by plotting and counting the simultaneous equations :—

(i) $\left.\begin{aligned} 2x - y &= 0 \\ x + y &= 0 \end{aligned}\right\}.$

(ii) $\left.\begin{aligned} 3x - 2y &= 10 \\ x + 5y &= -8 \end{aligned}\right\}.$

Verify by solving algebraically.

STATISTICS.

The graph of an algebraical equation in x and y shows the variations in one quantity y that depend upon and correspond to variations in another quantity x. We can, however, extend the use of graphs

Hours.	Temperature.
8 A.M.	48° F.
9 A.M.	50° F.
10 A.M.	53° F.
11 A.M.	60° F.
Noon.	68° F.
1 P.M.	75° F.
2 P.M.	77° F.
3 P.M.	75° F.
4 P.M.	67° F.
5 P.M.	60° F.
6 P.M.	58° F.
7 P.M.	50° F.
8 P.M.	45° F.

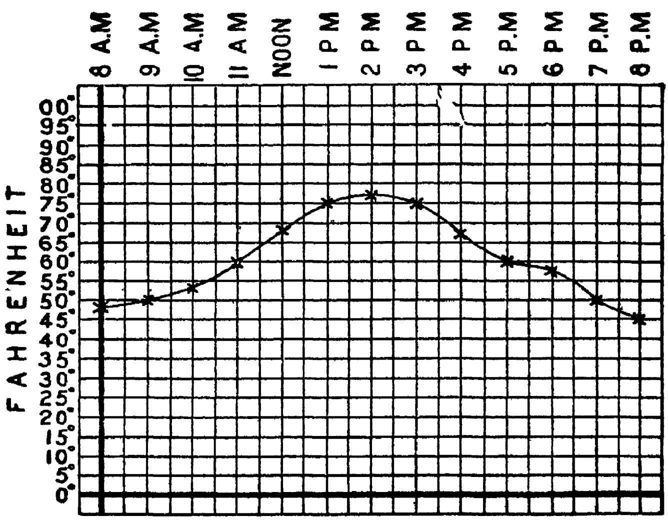


FIG. 144.

to illustrate the variations of *any* quantity that depend upon and correspond to variations in another quantity. Suppose, for ex-

ample, we wish to record the variations in temp rature during the course of a certain day. We can take as x-coordinates the numbers of hours (or half hours) that have elapsed since our first observation and as y-coordinates the temperatures in degrees Fahrenheit at the beginning of each hour (or half hour) respectively, and draw a continuous line through the points that represent the several observations. Thus Fig. 144 gives the graph that illustrates the data set forth in the foregoing table.

A graph that is based on observations made at short intervals may be taken to give fairly correct readings between two actual observations. Thus from Fig. 144 we may safely infer that the temperature at 10.30 A.M. was very nearly 56° F.

Exp. 158. Fig. 145 is a graph showing the variations in the speed of a train between two stations A and B, ten miles apart.

What was the speed 1, 3, $6\frac{1}{2}$, 9 miles from station A respectively? What was the maximum speed? How many miles from station B did it occur?

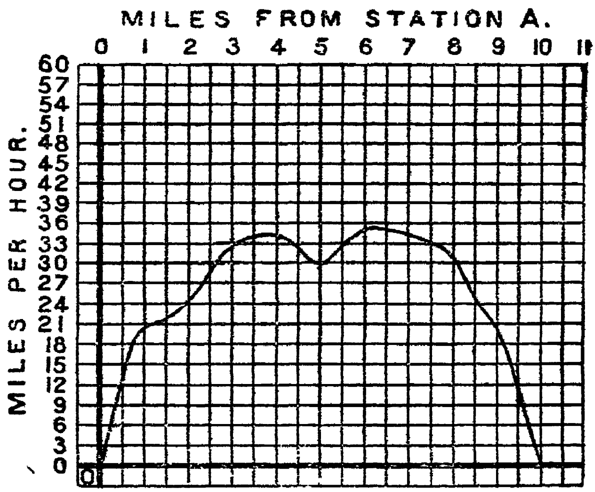


FIG. 145.

Exp. 159. Plot the graph showing how $\sin x$ varies as x increases from 0° to 360°.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin x$	0·00	0·50	0·86	1·00	0·86	0·50	0·00	– 0·50	– 0·86	– 1·00	– 0·86	– 0·50	0·00

MEASUREMENT OF AREAS.

Exp. 160. On any given unit of length describe a square; then the area of this square will be that unit of area that corresponds to the given unit of length.

Exp. 161. Describe the unit of area that corresponds to an inch

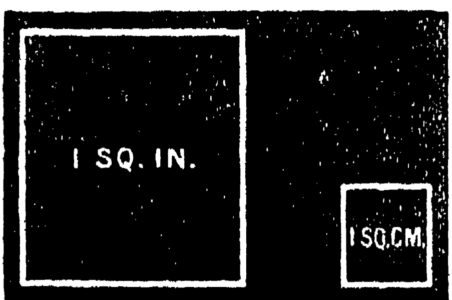


FIG. 146.

and call it a square inch. Also describe a square centimetre.

RECTANGLES.

Exp. 162. On a piece of squared paper draw a rectangle whose base measures 8 divisions and whose altitude measures 5 divisions.

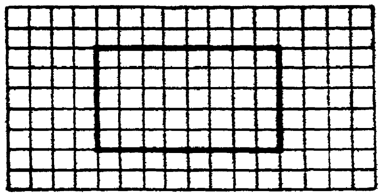


Fig. 147.

Call a division the unit of length. How many corresponding units of area does the rectangle contain?

Working :—

$$\left.\begin{array}{l}\text{No. of small squares}\\ \quad\text{in rectangle}\end{array}\right\} = \begin{array}{l}\text{No. of divisions in length} \times \text{No. of}\\ \quad\text{divisions in altitude.}\end{array}$$
$$= 8 \times 5$$
$$= 40$$

But each small square is the unit of area corresponding to the length of a division.

Exp. 163. On a piece of inch paper draw a square inch.

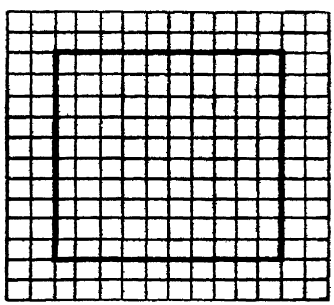


Fig. 148.

How many small squares are there in a square inch? What fraction of a square inch is each of the small squares?

From this Experiment we conclude that if $\frac{1}{10}$ inch is taken as the unit of length, $\frac{1}{100}(=\frac{1}{10} \times \frac{1}{10})$ square inch is the corresponding unit of area. Similarly, if $\frac{1}{100}$ inch is taken as the unit of length, $\frac{1}{10000}(=\frac{1}{100} \times \frac{1}{100})$ square inch is the corresponding unit of area.

Exp. 164. On a piece of inch paper draw a rectangle whose base measures 1·3 in. and whose altitude measures 0·7 in.

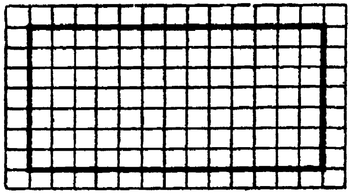


FIG. 149.

Find the area of the rectangle.

Working :—

Call $\frac{1}{10}$ in. the unit of length.
Then $\frac{1}{100}$ sq. in. is the corresponding unit of area.
Now the rectangle contains 13 × 7 units of area.

$$= \frac{13 \times 7}{100} \text{ sq. in.}$$
$$= 1{\cdot}3 \times 0{\cdot}7 \text{ sq. in.}$$
$$= 0{\cdot}91 \text{ sq. in.}$$

Exp. 165. On a piece of inch paper draw a rectangle whose base measures 1·26 in. and whose altitude measures 0·52 in.

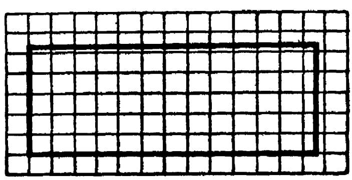


FIG. 150.

Find the area of the rectangle.

Working :—

Call $\frac{1}{100}$ in. the unit of length.
Then $\frac{1}{10000}$ sq. in. is the corresponding unit of area.
Now the rectangle contains 126 × 52 units of area.

$$= \frac{126 \times 52}{10000} \text{ sq. in.}$$
$$= 1{\cdot}26 \times 0{\cdot}52 \text{ sq. in.}$$
$$= 0{\cdot}6552 \text{ sq. in.}$$

From Experiments 162-165 we are led to conclude :—

The number of any linear unit in the base of a rectangle, multiplied by the number of the same linear unit in the altitude, gives the number of the corresponding square unit in the area.

Or, briefly :—

area of rectangle = base × altitude.

Learn this by heart.

Note.—This rule implies that the base and altitude of the rectangle have a common measure, that is to say, are *commensurable.* If the base and altitude are *incommensurable* we can express their measures in terms of any particular unit to as many decimal places as we desire, and then the rule will give the area of the rectangle to any required degree of accuracy. Thus, for the purposes of this Section, we may assume the rule to be true in all cases.

The abbreviation "rect. AB, AD," or "AB × AD," or "AB . AD" is used to denote "the rectangle having AB and AD for adjacent sides," and the rect. AB, AD is said to be "contained by" AB and AD. Similarly the abbreviation "sq. on AB," or "AB^2" is used to denote "the square having AB for side".

Exp. 166. Find the area of the rectangle PQ, PS drawn on inch paper.

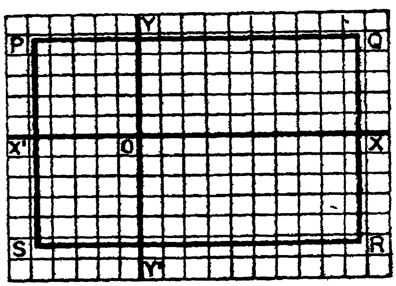


FIG. 151.

Working :—

$$PQ = (0{\cdot}47 + 0{\cdot}96) \text{ in.} = 1{\cdot}43 \text{ in.}$$
$$PS = (0{\cdot}47 + 0{\cdot}54) \text{ in.} = 1{\cdot}01 \text{ in.}$$
$$\therefore \text{area of rect. PQ, PS} = 1{\cdot}43 \times 1{\cdot}01 \text{ sq. in.}$$
$$= 1{\cdot}4443 \text{ sq. in.}$$
$$= 1{\cdot}4 \text{ sq. in. nearly.}$$

Note.—The result 1·4443 sq. in. is not reliable beyond the units digit. We cannot be sure, in practice, that we have estimated the

lengths PQ, PS correctly to the tenth of a division, and a discrepancy in either or both of these measurements might affect the second, or even the first, decimal place in the result 1·4443 sq. in.

Exp. 167. Find the area of the rectangle PQRS drawn on inch paper to the nearest tenth of a square inch.

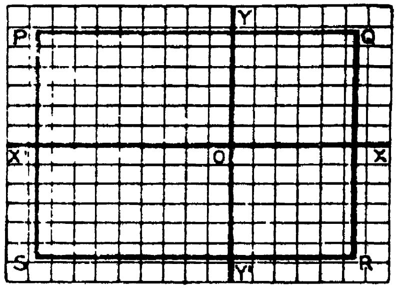


FIG. 152.

Exp. 168. Plot on inch paper the rectangles having the following points as vertices, and find the area of each rectangle :—

(i) (4, 7), (18, 7), (4, – 4), (18, – 4).
(ii) (12, 4), (12, – 9), (– 5, 4), (– 5, – 9).
(iii) (7, 3·2), (– 6, 3·2), (7, – 5), (– 6, – 5).
(iv) (– 13·7, – 3·6), (– 13·7, – 9·2), (– 5·9, – 3·6), (– 5·9, – 9·2).

Exp. 169. Draw on inch paper the rectangles having the following dimensions, and find the area of each rectangle :—

(i) 1·7 in. by 2·1 in.
(ii) 2·35 in. by 1·65 in.
(iii) 1·76 in. by 0·99 in.
(iv) 2·08 in. by 1·03 in.

Exp. 170. Find the areas of rectangles having the following dimensions :—

(i) 1 ft. 3 in. by 2 ft. 7 in.
(ii) 69 mm. by 38 mm.
(iii) 10 ft. by 100 in.
(iv) ½ mile by 3 yds. 2 ft. 9 in.
(v) x in. by y in.
(vi) a yds. by b ft.

Exp. 171. Find the areas of squares having the following sides :—

(i) 1 ft 7 in.
(ii) 2 yds. 2 ft. 9 in.
(iii) $3x$ in.
(iv) $(x + y)$ cm.

Exp. 172. Find the bases of the rectangles having the following areas and altitudes :—

(i) 16 sq. ft. 8 sq. in., 2 ft. 10 in.
(ii) 1 acre, 40 yds.
(iii) x sq. in., y in.
(iv) p sq. ft., q in.

Exp. 173. If the side of one square is 12 times the side of another square, show by a figure on squared paper that its area is 144 times as great. Hence prove that 1 sq. ft. = 144 sq. in.

Exp. 174. If 1 pole = $5\frac{1}{2}$ yds., prove as in Exp. 173 that 1 sq. pole = $30\frac{1}{4}$ sq. yds.

Exp. 175. Show by figures that the area of a rectangle is multiplied by 4, 9, 16, 25, a^2 if we multiply its linear dimensions by 2, 3, 4, 5, a.

Exp. 176. Find the sides of squares having the following areas :—

(i) 1 sq. ft 25 sq. in.
(ii) x sq. in.
(iii) $(a^2 + 2ab + b^2)$ sq. miles.

Exp. 177. Draw a rectangle and find its area (1) in sq. in., (2) in sq. cm., and hence find the number of square centimetres in a square inch.

Exp. 178. Repeat Exp. 177 with another rectangle of different dimensions.

Exp. 179. 1 in. = 2·54 cm. Verify, by calculation, the results you obtained in Exps. 177, 178.

Exp. 180. A carpet is required for a rectangular room 22 ft. by 18 ft., allowing a margin of 3 ft. all round. Draw a plan to scale R. F. $\frac{1}{72}$, and find the cost of the carpet at 3s. 4d. a sq. yd.

Exp. 181. A fence is required for a square enclosure containing 10 acres. Draw a plan to scale R. F. $\frac{1}{5280}$, and find the cost of the fencing at 2s. 8d. per yd.

Exp. 182. The area of a rectangle is 59 sq. yds. 1 sq. ft. 2 sq. in. and its perimeter is 31 yds. 6 in. Find its length and breadth.

Exp. 183. How many postage stamps measuring $\frac{23}{24}$ in. by $\frac{5}{8}$ in. will cover a sheet of paper 1 ft. 11 in. long and 1 ft. 3 in. wide?

RIGHT-ANGLED TRIANGLES.

Exp. 184. PQR in Fig. 153 is a right-angled triangle drawn on inch paper and PQRS is a rectangle on the same base and of the same altitude. Copy Fig. 153 on inch paper, and find the area of the triangle PQR.

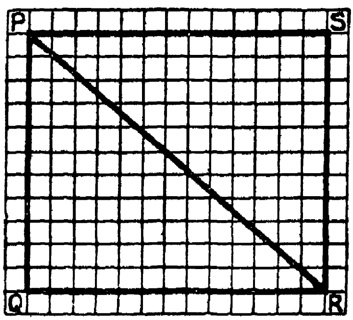


FIG. 153.

Working :—

$$\begin{aligned} \text{Area of } \triangle\, PQR &= \tfrac{1}{2} \text{ area of rect. } PQ, QR. \\ &= \tfrac{1}{2} \times 1{\cdot}1 \times 1{\cdot}3 \text{ sq. in.} \\ &= 0{\cdot}715 \text{ sq. in.} \end{aligned}$$

Exp. 185. Plot on inch paper the right-angled triangles having the following points for vertices, and find the area of each as in Exp. 184 :—

(i) (0, 0), (7, 0), (0, 11).
(ii) (14, −3), (−2, −3), (−2, 1).
(iii) (28·9, 1·7), (2·1, 1·7), (2·1, 9·5).
(iv) (9·1, 6·8), (−4·6, 6·8), (−4·6, −9·3).

PARALLELOGRAMS

Exp. 186. PQRS in Fig. 154 and in Fig. 155 is a parallelogram

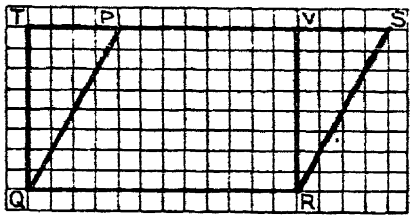


FIG. 154.

drawn on squared paper and TQRV is a rectangle on the same base

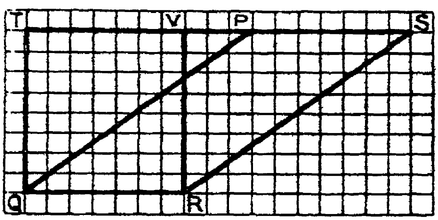


FIG. 155.

and of the same altitude. Copy Fig. 154 and Fig. 155 on squared paper, and show by counting divisions that △ TPQ = △ VSR.

Hence it follows that ▱gm. PQRS = rect. TQ, QR. Why?

Exp. 187. Repeat Exp. 186 two or three times, varying the base and altitude of the parallelogram each time.

Exp. 188. Copy Fig. 154 on plain, thin cardboard, and illustrate the result of Exp. 186 by cutting and superposing.

Exp. 189. Copy Fig. 155 on plain, thin cardboard, and illustrate the result of Exp. 186 by cutting and superposing.

Fig. 156 will show you how to do this. QY = RX, RM = QX, YZ is ∥ to PQ, MN is ∥ to RV.

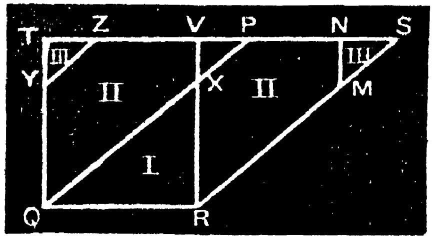


FIG. 156.

From Exps. 186-189 we are led to conclude :—

A parallelogram is equivalent to the rectangle on the same base and of the same altitude.

And hence,

The number of any linear unit in the base of a parallelogram multiplied by the number of the same linear unit in the altitude gives the number of the corresponding square unit in the area.

Or briefly, area of parallelogram = base × altitude.

And from this it follows that

Parallelograms on the same base or on equal bases and of the same altitude are equivalent.

Also,

Equivalent parallelograms on the same base or on equal bases are of the same altitude.

Learn these conclusions by heart.

Exp. 190. Illustrate the following truth by drawing a figure on thin, plain cardboard, cutting and superposing: *Parallelograms*

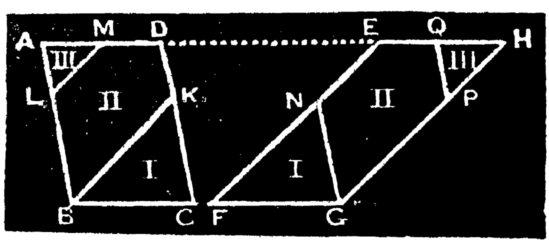


FIG. 157.

on the same base or on equal bases and of the same altitude are equivalent.

Fig. 157 will show you how to do this. BK, LM are parallel to FE; GN, PQ are parallel to CD; BL = CK; GP = FN.

Exp. 191. Plot on inch paper parallelograms having the following points as vertices, and find the area of each parallelogram :—

(i) (0, 1), (12, 1), (15, 8), (3, 8).
(ii) (10, 2), (5, −5), (−1, 2), (−6, −5).
(iii) (1, 8), (−9, 8), (−3, −5), (7, −5).
(iv) (0, 0), (−3, 0), (−8, −12), (−5, −12).

Exp. 192. Draw on inch paper parallelograms having the following dimensions, and find the area of each parallelogram :—

(i) base 1·4 in. altitude 0·8 in.
(ii) base 1·85 in. altitude 0·63 in.
(iii) base 2·01 in. altitude 1·11 in.
(iv) base 2·37 in. altitude 1·09 in.

Exp. 193. Find the areas of parallelograms having the following dimensions :—

(i) base 1 yd. 9 in.	altitude 2 ft. 8 in.
(ii) base 15·5 feet.	altitude 123 in.
(iii) base 1 mile 5 fur. 13 poles.	altitude $1\frac{1}{4}$ miles.
(iv) base x in.	altitude y ft.

Exp. 194. Find the bases of the following parallelograms :—

(i) area $7\frac{1}{2}$ acres altitude 5 chains.
(ii) area a sq. yds. altitude b ft.

Exp. 195. Draw a parallelogram ABCD. Draw EF the perpendicular distance between AD and BC. Draw GH the perpendicular distance between AB and DC. Find the area of the parallelogram (i) by taking BC as base and EF as altitude, (ii) by taking AB as base and GH as altitude.

Compare the two results. What is the average result?

Exp. 196. Construct parallelograms ABCD from the following data :—

(i) AB = 1·6 in., AD = 1·1 in., $\angle$ A = 72°.
(ii) AB = 3·5 cm., AC = 4·6 cm., BD = 2·8 cm.
(iii) AB = 1·2 in., BC = 0·9 in., AC = 1·9 in.
(iv) AC = 2·9 cm., $\angle$ BAC = 29°, $\angle$ BCA = 46°.

Find the area of each parallelogram by two independent calculations, drawing such lines and making such measurements as are necessary. Write down the average result in each case.

Exp. 197. Construct parallelograms ABCD from the following data :—

(i) AB = 1·2 in., $\angle$ A = 38°, area = 1·08 sq. in.
(ii) AB = 3·4 cm., $\angle$ D = 112°, area = 9·18 sq. cm.

ANY TRIANGLES.

Exp. 198. PQR in Fig. 158 is a triangle drawn on squared paper, and SQRT is a rectangle on the same base and of the same altitude. Copy Fig. 158 on squared paper, and show by counting divisions that

$$\triangle\, SPQ + \triangle\, TPR = \tfrac{1}{2} \text{ rect. } SQ, QR,$$

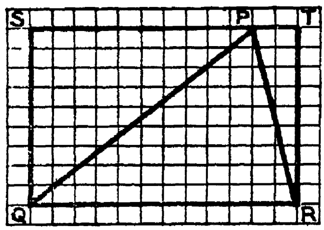


FIG. 158.

and, therefore, $\triangle$ PQR = $\frac{1}{2}$ rect. SQ, QR. Why?

Exp. 199. PQR in Fig. 159 is a triangle drawn on squared paper, and SQRT is a rectangle on the same base and of the same altitude. Copy Fig. 159 on squared paper, and show by counting divisions that

$$\triangle\, SPQ - \triangle\, TPR = \tfrac{1}{2} \text{ rect. } SQ, QR.$$

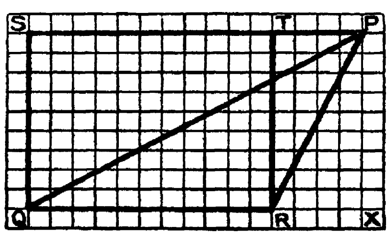


FIG. 159.

But $\triangle$ SPQ = $\triangle$ PQX and $\triangle$ TPR = $\triangle$ PRX.
Therefore $\triangle$ PQR = $\frac{1}{2}$ rect. SQ, QR. Why?

Exp. 200. Repeat Exps. 198, 199 two or three times, varying the base and altitude of the triangle each time.

Exp. 201. Copy Fig. 158 on plain, thin cardboard, and illustrate the result of Exp. 198 by cutting and superposing.

Exp. 202. Copy Fig. 159 on plain, thin cardboard, and illustrate the result of Exp. 199 by cutting and superposing.

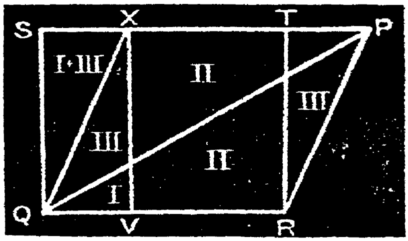


FIG. 160.

Fig. 160 will show you how to do this. **QX** is ‖ to **RP** and **XV** ‖ to **SQ**.

From Exps. 198-202 we are led to conclude :—

A triangle is equivalent to half the rectangle on the same base and of the same altitude.

And hence,

The number of any linear unit in the base of a triangle multiplied by half the number of the same linear unit in the altitude gives the number of the corresponding square unit in the area.

Or briefly,

$$\text{area of triangle} = \tfrac{1}{2}\text{ base} \times \text{altitude}.$$

And from this it follows that :—

Triangles on the same base or on equal bases and of the same altitude are equivalent.

Also,

Equivalent triangles on the same base or on equal bases are of the same altitude.

Also,

A triangle is equivalent to half a parallelogram on the same base and between the same parallels.

Learn these conclusions by heart.

Exp. 203. Illustrate the following truth by drawing a figure on thin, plain cardboard, cutting and superposing: *A triangle is*

equivalent to half a parallelogram on the same base and between the same parallels.

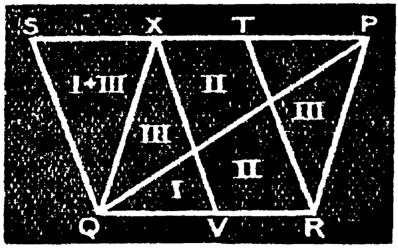


FIG. 161.

Fig. 161 will show you how to do this. QX is || to RP and XV || to SQ.

Exp. 204. Plot on inch paper triangles having the following points as vertices, and find the area of each triangle :—

(i) (2, 2), (11, 12), (14, 2).
(ii) (7, −3), (−4, 4), (−8, −3).
(iii) (6, 1), (10, 10), (−3, 1).
(iv) (−1, 7), (−16, 0), (−12, 7).

Exp. 205. Draw on inch paper triangles having the following dimensions, and find the area of each triangle :—

(i) base 1·5 in. altitude 0·9 in.
(ii) base 1·7 in. altitude 1·1 in.
(iii) base 2·05 in. altitude 0·85 in.
(iv) base 1·37 in. altitude 1·13 in.

Exp 206. Find the areas of triangles having the following dimensions :—

(i) base 2 ft. 7 in. altitude 1 ft. 6 in.
(ii) base 120 ft. altitude 27 yds.
(iii) base $\frac{3}{4}$ mile. altitude 3 fur. 27 poles.
(iv) base p ft. altitude q in.

Exp. 207. Find the altitudes of the following triangles :—

(i) area 6·39375 acres base 13 chains 75 links.
(ii) area 52 sq. yds. 18 sq. in. base 8 yds. 2 ft. 3 in.

Exp. 208. Find the bases of the following triangles :—

(i) area 6·42096 acres altitude 936 links.
(ii) area $\sqrt{2}$ sq. yds. altitude 1 ft.

Exp. 209. Draw a triangle ABC. Draw AD, BE, CF perpendicular to BC, CA, AB respectively. Find the area of the triangle (i) by taking BC as the base and AD as the altitude ; (ii) by taking

CA as the base and BE as the altitude; (iii) by taking AB as the base and CF as the altitude.

Compare the three results. What is the average result?

Exp. 210. Construct triangles ABC from the following data:—

(i) $\angle A = 22°$, $\angle C = 49°$, $BC = 4{\cdot}12$ cm.

(ii) $\angle C = 87°$, $BC = 1{\cdot}34$ in., $CA = 1{\cdot}73$ in.

Find the area of each triangle by three independent calculations, drawing such lines and making such measurements as are necessary. Write down the average result in each case.

Exp. 211. Construct triangles ABC from the following data:—

(i) $AB = 1{\cdot}1$ in. $BC = 1{\cdot}9$ in., area $= 0{\cdot}5$ sq. in.

(ii) $BC = 2{\cdot}9$ cm., $\angle C = 49°$, area $= 6$ sq. cm.

Exp. 212. Draw three different quadrilaterals, but let the diagonals of each be at right angles to one another and measure 1·75 in. and 2·15 in. respectively. Calculate the area of each quadrilateral and compare the results.

Exp. 213. The diagonals of a quadrilateral measure a in. and b in. respectively and they cut one another at right angles. Show with the help of a figure that the area of the quadrilateral is $\frac{1}{2}ab$ sq. in.

Exp. 214. The parallel sides of a trapezium measure a in. and b in. respectively and the height h in. Show with the help of a figure that the area of the trapezium is $\frac{1}{2}(a + b)h$ sq. in.

Draw a diagonal and consider the trapezium as the sum of two triangles of the same altitude, h in., and of bases, a in. and b in. respectively.

Exp. 215. A diagonal of a quadrilateral measures d in. and the perpendiculars upon it from the outlying angular points a in. and b in. respectively. Show with the help of a figure that the area of the quadrilateral is $\frac{1}{2}d(a + b)$ sq. in.

Consider the quadrilateral as the sum of two triangles on the same base of length d in., but on opposite sides of it and of altitudes a in. and b in. respectively.

Exp. 216. Construct quadrilaterals ABCD from the following data:—

(i) $AB = 3{\cdot}2$ cm.; $BC = 2{\cdot}9$ cm.; $CD = 4{\cdot}1$ cm.; $DA = 3{\cdot}4$ cm.; $AC = 5{\cdot}9$ cm.

(ii) $AB = 0{\cdot}7$ in.; $BC = 1{\cdot}1$ in.; $CD = 1{\cdot}6$ in.; $DA = 1{\cdot}2$ in.; $\angle ABC = 75°$.

(iii) $AB = 1{\cdot}9$ in.; $AD = DC = 1{\cdot}2$ in.; $\angle A = 62°$; DC is $\parallel$ to AB.

(iv) $AC = 2{\cdot}03$ in.; $BD = 1{\cdot}87$ in.; AC and BD bisect each other at right angles.

(v) $AC = 4{\cdot}6$ cm.; $BD = 5{\cdot}2$ cm.; AC and BD bisect each other at 65°.

Find the area of each quadrilateral, drawing such lines and making such measurements as are necessary.

THE COMPLEMENTS OF THE PARALLELOGRAMS ABOUT A DIAGONAL OF A PARALLELOGRAM.

Exp. 217. ABCD in Fig. 162 is a parallelogram. FG, HK are parallels to the sides through a point E on the diagonal BD.

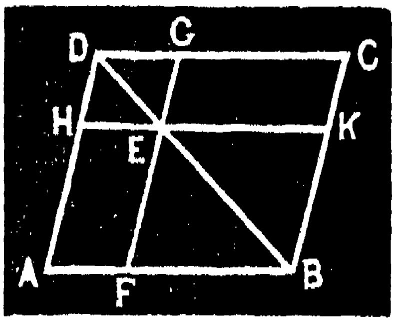


FIG 162.

HEGD, FBKE are called the parallelograms about the diagonal BD and AFEH, EKCG are called their complements. Illustrate by measurement and calculation the following truth:—

The complements of the parallelograms about a diagonal of any parallelogram are equivalent.

Learn this by heart.

ANY RECTILINEAL FIGURES.

Exp. 218. PQR in Fig. 163 is a triangle drawn on inch paper. Copy Fig. 163 on inch paper, and find the area of the triangle PQR by counting divisions.

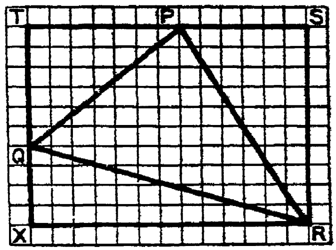


FIG. 163.

Working :—

$$\triangle PQR = \text{rect. } TX, XR - \triangle QXR - \triangle RSP - \triangle PTQ.$$
$$= 0{\cdot}53 \text{ sq. in.}$$

Exp. 219. PQRS in Fig. 164 is a square drawn on inch paper. Copy Fig. 164 on inch paper, and find the area of the square PQRS by counting divisions.

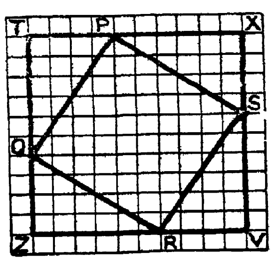


FIG. 164.

Working :—

$$\text{Square } PQRS = \text{rect. } TZ, ZV - \triangle PTQ - \triangle QZR - \triangle RVS - \triangle SXP.$$
$$= 0{\cdot}52 \text{ sq. in.}$$

Exp. 220. PQR in Fig. 165 is a triangle drawn on inch paper. Copy Fig. 165 on inch paper, and find the area of the triangle PQR by counting divisions.

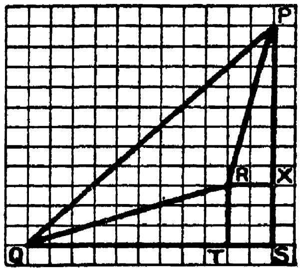


Fig. 165.

Working :—

$\triangle$ PQR = $\triangle$ PQS − $\triangle$ RQT − $\triangle$ PRX − rect. RT, TS.
= 0·33 sq. in.

Exp. 221. PQRSTX in Fig. 166 is a rectilineal figure drawn on inch paper. Copy Fig. 166 on inch paper, and find the area of the rectilineal figure PQRSTX by counting divisions.

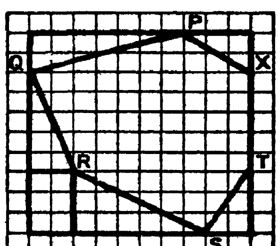


Fig. 166.

Exp. 222. Copy Figs. 167, 168 on inch paper, and find the areas of the rectilineal figures by adding and subtracting areas of right-angled triangles and rectangles as in Exps. 218-221.

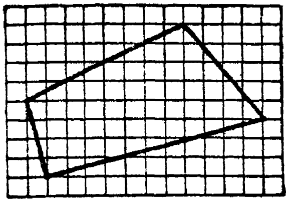


FIG. 167.

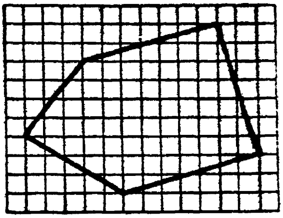


FIG. 168.

Exp. 223. Plot on inch paper the rectilineal figures having the following points for vertices, and find the area of each by adding and subtracting areas of right-angled triangles and rectangles as in Exps. 218-222 :—

(i) (2, 3), (7, −6), (−7, 1).
(ii) (0, 0), (3, 8), (12, 12).
(iii) (2, 9), (9, 3), (−2, −6), (−4, 2).
(iv) (8, 1), (3, 7), (−3, 9), (−8, 4).

THE THEOREM OF PYTHAGORAS.

Exp. 224. ABC in Fig. 169 is a right-angled triangle drawn on squared paper and ACDE, ABGF, BCKH are squares described

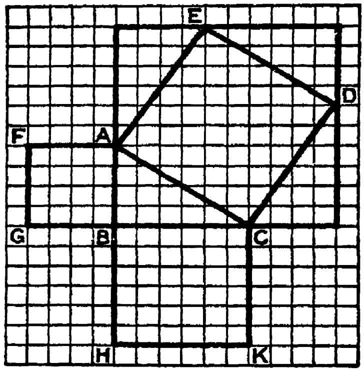


Fig. 169.

upon its hypotenuse AC and its sides AB, BC respectively. Copy Fig. 169 on squared paper, and show by counting divisions that

Square on AC = square on AB + square on BC.

Exp. 225. Repeat Exp. 224 with three right-angled triangles, altering the lengths of the sides containing the right angle in each case.

Exp. 226. Illustrate the result of Exp. 224 by drawing a figure on plain thin cardboard, cutting and superposing.

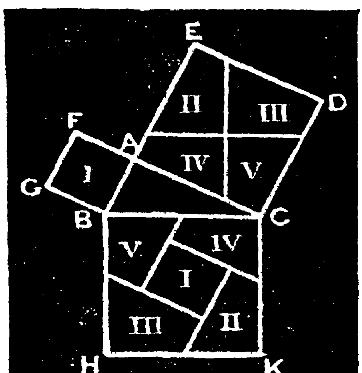


Fig. 170.

Fig. 170 will show you how to do this. Through the middle point of ACDE, one of the smaller squares, one straight line is drawn ‖ to, and another ⊥ to the hypotenuse BC, thus dividing the square into four pieces, which, together with the square ABGF, can be made to fit exactly into the square BCKH (Perigal's dissection).

Exp. 227. Repeat Exp. 226 with another right-angled triangle, altering the lengths of AB and AC.

Exp. 228. ABC is a right-angled triangle and BCKH is a square described on BC. CD is drawn perpendicular to AC meeting HK

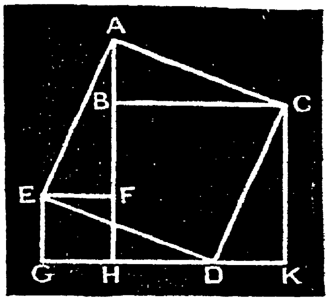


FIG. 171.

in D, and the rectangle ACDE is completed. ACDE will be found to be a square.

Now EF is drawn perpendicular to BH, and the rectangle EFHG is completed. EFHG will be found to be a square and FH equal to AB.

Copy Fig. 171 on thin cardboard, cut out the pieces EGD, DKC, and show that they can be made to fit exactly into the space EFBCA.

From Exps. 224-228 we are led to conclude :—

In a right-angled triangle the square on the hypotenuse is equivalent to the sum of the squares on the sides containing the right angle.

Or, briefly, in a right-angled triangle,

$$(\text{hypotenuse})^2 = (\text{base})^2 + (\text{perpendicular})^2.$$

And hence,

$$\text{hypotenuse} = \sqrt{(\text{base})^2 + (\text{perpendicular})^2}.$$

Learn these conclusions by heart.

Exp. 229. Plot the following points on inch paper and calculate to the nearest hundredth of an inch their distances from the origin of coordinates. Verify by measurements :—

(i) (13, 7).
(ii) (−6, 11).
(iii) (−13, −6).
(iv) (15, −7).

Fig. 172 shows an easy way of measuring the distance on squared

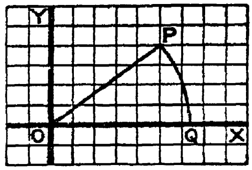


FIG. 172.

paper of any point P from the origin of coordinates. With centre O and radius OP describe an arc cutting OX at Q. Then OP = OQ.

Exp. 230. Plot the following pairs of points on inch paper and calculate to the nearest hundredth of an inch their distances from each other. Verify by measurements :—

(i) (3, 6), (16, 12).
(ii) (12, 6), (−7, 9).
(iii) (2, 13), (−9, −2).
(iv) (16, −6), (−4, 2).

Fig. 173 shows an easy way of measuring the distance on squared paper of any point P from any other point Q. With centre P and

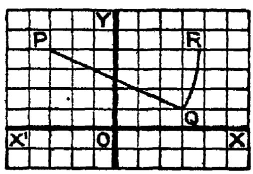


FIG. 173.

radius PQ describe an arc cutting the horizontal through P at R. Then PQ = PR.

Exp. 231. Plot the triangle whose vertices are (13, 11), (6, −8), (−7, 6), and ascertain by calculation whether it is equilateral, isosceles or scalene.

Exp. 232. Plot the points (−2, 6), (6, 2), (6, −2), and show by calculation that they lie on the circumference of a circle having (0, 0) for centre. Verify by describing the circle.

Exp. 233. Plot the points (2, 1), (10, −3), (10, −7), and show by calculation that they lie on the circumference of a circle having (4, −5) for centre. Verify by describing the circle.

Exp. 234. Plot the points (7, 2), (4, 9), (0, 9), and find by calculation the centre of the circle on which they lie. Verify by describing the circle.

Exp. 235. Construct a right-angled triangle whose base and perpendicular are 4·7 cm. and 3·6 cm. respectively. Calculate the length of its hypotenuse to the nearest hundredth of a cm., and verify by actual measurement.

Exp. 236. Construct a right-angled triangle whose base and hypotenuse are 34 mm. and 51 mm. respectively. Calculate the length of its perpendicular to the nearest mm., and verify by actual measurement

Exp. 237. Show by calculation that the triangle whose sides are 1 mile 5 fur. 12 poles, 4 fur. 5 poles, 1 mile 5 fur. 37 poles is right angled.

Exp. 238. Show by a diagram that if the side of a square measures a in., its diagonal measures $a\sqrt{2}$ in.

Exp. 239. Construct a square of side 1 in. and calculate the length of a diagonal to the nearest hundredth of an inch. Verify by actual measurement.

Exp. 240. Show by a diagram that if the diagonal of a square measures a in., its side measures $\frac{a}{\sqrt{2}}$ in.

Exp. 241. Construct a square of diagonal 5 cm. and calculate the length of a side to the nearest hundredth of a centimetre. Verify by actual measurement.

Exp. 242. Show by a diagram that if the side of an equilateral triangle measures a in., its altitude measures $\frac{a\sqrt{3}}{2}$ in.

Exp. 243. Construct an equilateral triangle of side 1·75 in. and calculate the length of its altitude to the nearest hundredth of an inch. Verify by actual measurement.

Exp. 244. Construct a triangle ABC from the following data: AB = 1·3 in., BC = 2·1 in., BD = 0·4 in., where D is the foot of the perpendicular from A on BC. Calculate AD, and find the area of the triangle.

Exp. 245. Prove that in any triangle ABC

$$\mathrm{BD} = \frac{c^2 + a^2 - b^2}{2a} \text{ linear units,}$$

where BC = a units, CA = b units, AB = c units, and D is the foot of the perpendicular from A on BC.

For $\mathrm{AB}^2 - \mathrm{BD}^2 = \mathrm{AD}^2 = \mathrm{AC}^2 - \mathrm{CD}^2 = \mathrm{AC}^2 - (\mathrm{BC} - \mathrm{BD})^2$.

$\therefore$ if BD = x units, $c^2 - x^2 = b^2 - (a^2 + x^2 - 2ax)$.

Exp. 246. Construct a triangle ABC from the following data: AB = 3·8 cm., BC = 4·1 cm., CA = 3·7 cm. Draw AD perpendicular to BC, and calculate BD. Hence find AD and the area of the triangle.

Exp. 247. Prove that in any triangle ABC

$$AD = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a} \text{ linear units,}$$

where BC = a units, CA = b units, AB = c units, $s = \dfrac{a+b+c}{2}$ and D is the foot of the perpendicular from A on BC.

For $AD^2 = AB^2 - BD^2$ and $BD = \dfrac{c^2+a^2-b^2}{2a}$ units.

$$\therefore AD^2 = \left(c - \frac{c^2+a^2-b^2}{2a}\right)\left(c + \frac{c^2+a^2-b^2}{2a}\right) \text{ sq. units}$$

$$= \frac{b^2-(c-a)^2}{2a} \times \frac{(c+a)^2-b^2}{2a} \text{ sq. units}$$

$$= \frac{(b-c+a)(b+c-a)}{2a} \times \frac{(c+a-b)(c+a+b)}{2a} \text{ sq. units}$$

$$= \frac{2(s-c)\,.\,2(s-a)}{2a} \times \frac{2(s-b)\,.\,2s}{2a} \text{ sq. units.}$$

Exp. 248. Prove that in any triangle ABC

area $= \sqrt{s(s-a)(s-b)(s-c)}$ square units,

where the letters have the same significance as in Exp. 247.

Exp. 249. Find the areas of triangles whose sides are :—

(i) 51 yds., 37 yds., 20 yds.
(ii) 25 miles, 17 miles, 12 miles.
(iii) 11 ft., 10 ft. 5 in., 3 ft. 1 in.
(iv) 11 yds. 2 ft. 5 in., 24 yds. 3 in., 34 yds. 2 ft. 10 in.

CIRCLES.

CHORDS AND ARCS OF CIRCLES.

Exp. 250. Describe a circle and draw a straight line joining any two points on the circumference. Call it a chord of the circle.

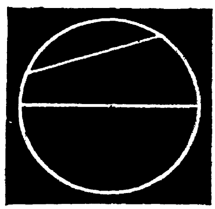


FIG. 174.

Draw a chord passing through the centre. Call it a diameter of the circle.

Exp. 251. In a circle of radius 22 mm. place, if possible, chords of the following lengths : 2·8 cm., 33 mm., 57 mm., 7·3 mm. What length is the longest chord that can be placed in this circle ?

SYMMETRY.

Exp. 252. Fold a sheet of paper once and cut away with your scissors any irregular portion of the folded sheet bounded on one side by the crease. Open it out and you will obtain a figure

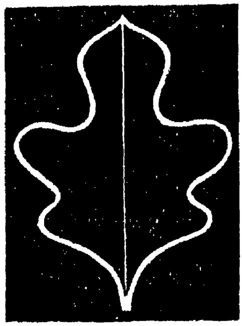


FIG. 175.

divided by the crease into two halves that exactly balance one another. Such a figure is said to be *symmetrical*, and the line of division is called the *axis of symmetry*.

Exp. 253. Illustrate by drawing on tracing paper and by folding the following truth :—

A circle is symmetrical about any diameter.

Learn this by heart.

Exp. 254. Draw two circles on tracing paper and show by folding that the straight line passing through their centres (called their line of centres) is an axis of symmetry of the figure.

Exp. 255. Fold a sheet of paper once and find, by pricking with a pin, two or three pairs of points that coincide when the sheet is

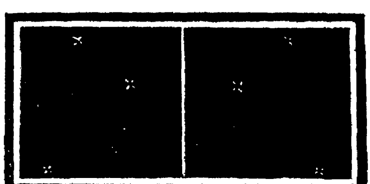


FIG. 176.

folded. Open it out and you will obtain pairs of points that are

said to be symmetrically opposite with regard to the axis of symmetry

Exp. 256. Describe a circle on tracing paper, draw a diameter, and find, by folding and pricking with a pin, two points on the

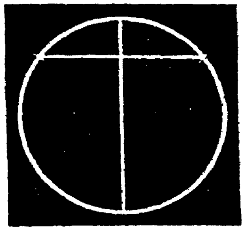


FIG. 177.

circumference symmetrically opposite with regard to that diameter. Show that the diameter bisects at right angles the chord joining the points.

Exp. 257. Illustrate by drawing figures and by measurements with ruler and protractor the following truth and its converse:—

> *A straight line drawn from the centre of a circle to bisect a chord which is not a diameter is at right angles to the chord.*

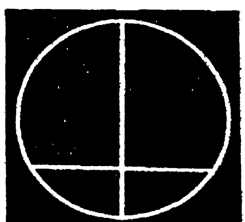


FIG. 178.

Learn this by heart.

Exp. 258. In a circle of radius 1 in. place a chord of length 1·6 in. Find by calculation and verify by measurement the distance of this chord from the centre of the circle.

Exp. 259. In a circle of diameter 1·4 in. place a chord of length 1 in. Find by calculation to the hundredth of an inch, and verify by measurement the distance of this chord from the centre of the circle.

Exp. 260. In a circle of radius 6·5 cm. place a chord distant 6 cm. from the centre. Find by calculation and verify by measurement the length of the chord.

Exp. 261. Describe a circle having a chord of length 1·4 in. distant 2·4 in. from the centre.

Exp. 262. In a circle of diameter 2·6 in. place a chord of length 1·8 in. Find by calculation and verify by measurement the distances of the ends of the chord from the ends of the diameter bisecting the chord.

Exp. 263. In a circle of radius 2·3 cm. are placed two parallel chords of lengths 4·2 cm. and 3·6 cm. respectively (1) on the same side of the centre, (2) on opposite sides of the centre. Draw a rough diagram, and calculate to the nearest hundredth of a centimetre the distance between the chords.

Exp. 264. Any part of the circumference of a circle is called an arc, and the perpendicular from the middle point of an arc upon

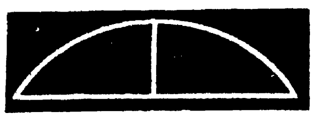


FIG. 179.

the chord joining its extremities is called the **height of the arc.** From a circle of radius 1·3 in. cut off an arc of height 0·8 in. Find by calculation and verify by measurement the chord of the arc.

Exp. 265. All circles through the points (a, b) and $(-a, b)$ have their centres on the axis of Y. Give reasons for this, and illustrate your answer with a diagram.

Exp. 266. In a circle of radius a in. a chord equal in length to the radius is distant $\frac{a\sqrt{3}}{2}$ in. from the centre. Give reasons for this, and illustrate your answer with a diagram.

Exp. 267. In a circle of radius r cm. a chord of length $2a$ cm. is distant $\sqrt{r^2 - a^2}$ cm. from the centre. Give reasons for this, and illustrate your answer with a diagram. Hence show that the mid-points of equal chords in a circle lie on another circle having the same centre.

Circles having the same centre are called **concentric circles.**

Exp. 268. Draw any straight line AB and its perpendicular bisector CD. Take any point E on CD and join EA and EB. Why

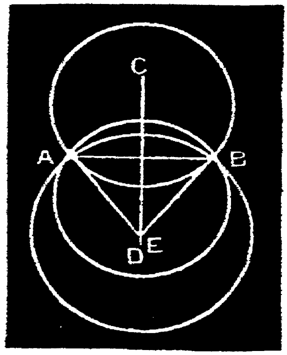


FIG. 180.

is EA = EB? With centre E describe a circle to pass through A and B. Describe two other circles to pass through A and B. Must they both have their centres on CD?

AB is called a common chord of these three circles.

Exp. 269. Show by drawing on tracing paper and folding that

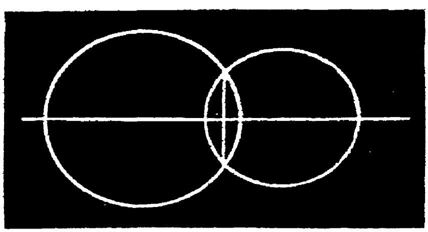


FIG. 181.

the line of centres of two intersecting circles is the perpendicular bisector of their common chord. Notice that this same conclusion can be drawn from Exp. 268.

Exp. 270. Describe two circles of radii 1·85 cm. and 2·55 cm. respectively and having a common chord of length 1·5 cm. Calculate the distance between their centres, and verify by measurement.

Exp. 271. Take any three points A, B and C not in a straight line. Draw the perpendicular bisectors of the lines joining A to B

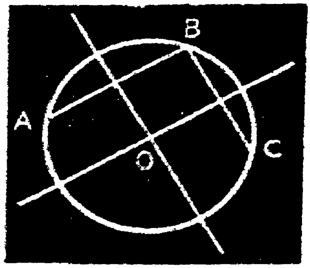


FIG. 182.

and B to C. Let them meet at O. Why is O equidistant from A, B and C? Why is O the *only* point equidistant from A, B and C? With centre O describe a circle passing through A, B and C.

From this experiment we are led to conclude:—

There is one circle and one only which passes through three given points not in a straight line.

Learn this by heart.

Exp. 272. Describe circles on inch paper to pass through the following sets of points, and measure, as in Exp. 230, the radius in each case to the nearest hundredth of an inch.

(i) $(1, 8)$, $(13, -2)$, $(12, 12)$.
(ii) $(10, 7)$, $(4, -11)$, $(-3, 9)$.
(iii) $(1, 5)$, $(-1, 6)$, $(-6, 4)$.
(iv) $(-8, 0)$, $(-5, -9)$, $(-1, -11)$.

Exp. 273. Make a triangle ABC having AB = 1·1 in., the angle ACB = 73° and the angle ABC = 65°. Describe a circle passing through all its vertices. Call it the circum-circle of the triangle.

Exp. 274. Illustrate, by drawing figures and by cutting out and superposing, the following truths and their converses:—

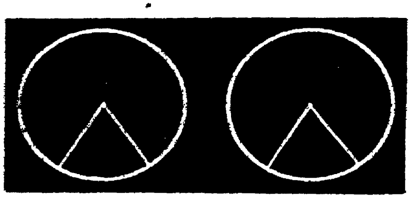


FIG. 183.

In equal circles (or in the same circle) if two arcs subtend (that is are opposite to) equal angles at the centres they are equal.

And,

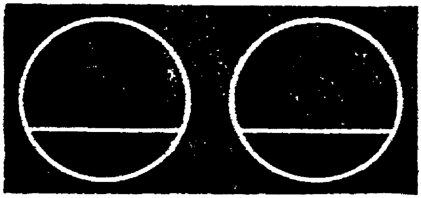


FIG. 184.

In equal circles (or in the same circle) if two chords are equal they cut off equal arcs.

Learn these by heart.

Exp. 275. If an arc of a circle subtends double the angle at the centre that another arc of the same circle subtends, is the first arc double the second? Illustrate your answer by drawing a figure, cutting out and superposing.

Exp. 276. If a chord of a circle is double another chord of the same circle, is the arc cut off by the first chord double the arc cut off by the second chord? Illustrate your answer by drawing a figure, cutting out and superposing.

Exp. 277. Describe a circle and divide its circumference into (1) four (2) three equal parts by making equal angles at the centre.

Exp. 278. Describe a circle and place in it a chord equal in length to the radius. What fraction of the whole circumference is the arc cut off by this chord?

Exp. 279. Using the result of Exp. 278, place a regular hexagon inside a circle of radius 1 in. Explain your construction. What length is each side of the hexagon? Verify by measurement.

Exp. 280. Place a square inside a circle of radius 18 mm. Explain your construction. Calculate the length of each side of the square to the tenth of a millimetre, and verify by measurement.

Exp. 281. Place an equilateral triangle inside a circle of radius 1 in. Explain your construction. Calculate the length of each side of the triangle to the hundredth of an inch, and verify by measurement.

MEASUREMENT OF CIRCUMFERENCE.

Exp. 282. Measure as accurately as you can the circumferences of three or four circular objects. Let them be of different sizes such as a coin, a lead pencil, a watch, a round box. Note the measurements. Now measure their diameters, and find for each object the value of the quotient $\frac{\text{circumference}}{\text{diameter}}$. It can be proved that, with perfect accuracy, this quotient will work out the same in each case and will be found to be 3·14 as far as the second decimal place. But the circumference and diameter of a circle can be shown by mathematical reasoning to be incommensurable, that is to say, to have no common measure, however small, and so the quotient $\frac{\text{circumference}}{\text{diameter}}$ can only be approximately expressed as a decimal fraction. It is denoted by the Greek letter π, and its value to 7 places of decimals is 3·1415926. For practical purposes π is often taken equal to 3·1416 or even $3\frac{1}{7}$.

A little ingenuity will suggest various methods of measuring the circumferences of circular objects. For a coin, put a small spot of ink on the rim and roll the coin in a straight line across a sheet of paper; then measure the distance between the ink marks on the paper. For a pencil, wind it round and round with thread and then divide the length of thread used by the number of rounds. Take care that the rounds touch one another but do not overlap. For a round box, wrap it tightly round with a strip of paper and prick with a pin through the double thickness of the paper where it overlaps; then unwrap and measure the distance between the two pin-pricks.

From this experiment we are led to conclude :—

$$\text{circumference of circle} = \text{diameter} \times \pi.$$

And hence,

$$\text{diameter of circle} = \text{circumference} \div \pi.$$

Learn these conclusions by heart.

Exp. 283. Taking $\pi = \frac{22}{7}$ calculate the circumferences of the following circles :—

(i) diameter 49 in.
(ii) diameter 28 miles.
(iii) radius 56 mm.

Exp. 284. Taking $\pi = \frac{22}{7}$ calculate the diameters of the following circles :—

(i) circumference 44 ft.
(ii) circumference 11 metres.
(iii) circumference 1540 miles.

Exp. 285. Calculate to the thousandth of an inch the circumference of a circle whose diameter measures 19 in.

Exp. 286. Calculate to the thousandth of a millimetre the diameter of a circle whose circumference measures 15 cm.

Exp. 287. Taking $\pi = \frac{22}{7}$ calculate the number of revolutions a wheel will make in travelling half a mile if its diameter measures 28 in.

Exp. 288. Taking $\pi = \frac{22}{7}$, and assuming that the radius of the earth is 4000 miles, calculate how long it would take a man to travel round the equator at an average rate of 10 miles an hour.

MEASUREMENT OF AREA.

Exp. 289. Describe a circle and draw a number of diameters dividing it into equal parts. Such parts are called **sectors.** The greater the number of sectors, the less does each sector differ in

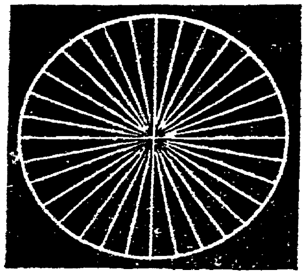


FIG. 185.

shape from a triangle, and, when the number is very great indeed, we may take the area of each sector as equal to the area of a triangle whose base is equal to the arc of the sector and whose height is the radius of the circle.

But area of circle = sum of areas of these exceedingly small sectors.

Therefore area of circle = rect. contained by $\frac{1}{2}$ radius and sum of arcs of sectors.

= rect. contained by $\frac{1}{2}$ radius and circumference of circle.

= rect. contained by $\frac{1}{2}$ radius and ($\pi \times$ diameter).

= πr^2 sq. units (where r linear units = radius).

And hence,

$$r^2 \text{ sq. units} = \frac{\text{area}}{\pi}.$$

$$\text{Therefore sq. on radius} = \frac{\text{area}}{\pi}.$$

$$\text{And radius of circle} = \sqrt{\frac{A}{\pi}} \text{ linear units (where A sq. units = area).}$$

Learn these conclusions by heart.

Exp. 290. Describe a quarter of a circle of 1 in. radius on inch paper, and find its approximate area by counting squares according to the following rule :—

Count a broken square as 1, 0 or ½ according as it appears greater than, less than or equal to half a complete square.

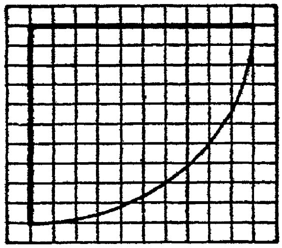


FIG. 186.

True area = ¼ of $\pi \times r^2$ sq. in. = 0·785 sq. in. (correct to 3 places decimals) since $r = 1$.

The greater the number of squares contained by the figure, the

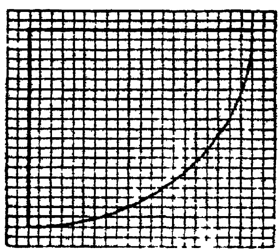


FIG. 187.

more nearly does the result obtained by this rule approach the true area. Apply the rule, for example, to Fig. 187 and compare your results.

Exp. 291. Describe a circle of 0·8 in. radius on inch paper, and find its approximate area by counting squares. True area = $\pi \times r^2$ sq. in. = 2·010 sq. in. (correct to 3 places of decimals) since $r = 0{\cdot}8$.

Exp. 292. Calculate the area of a circle whose radius measures 2 yds. 1 ft. 7 in. ($\pi = \frac{22}{7}$).

Exp. 293. Calculate to the tenth of a yard the radius of a circle whose area measures 1 acre

Exp. 294. Calculate the area of a circle in acres and poles whose circumference measures $60\frac{1}{2}$ chains ($\pi = \frac{22}{7}$).

Exp. 295. A plane circular ring is bounded by two concentric circles of radii R ft. and r ft. respectively. Show with the help of a diagram that the area of the ring $= \pi(R - r)(R + r)$ sq. ft.

Exp. 296. Find the breadth of a plane circular ring which is bounded by two concentric circles of areas 154 sq. in. and 308 sq. in. respectively ($\pi = \frac{22}{7}$).

Exp. 297. A circular grass plot whose diameter is 40 yds. contains a gravel path 1 yd. wide running round it 1 yd. from the edge; find what it will cost to turf the grass plot at 4d. per sq. yd. ($\pi = \frac{22}{7}$).

Exp. 298. What will be the expense of paving a circular court of 30 ft. diameter at 2s. 3d. per sq. ft., leaving in the centre a regular hexagonal space of $3\frac{1}{2}$ ft. side? ($\pi = \frac{22}{7}$).

Exp. 299. In cutting 4 equal circles, the largest possible, out of a piece of cardboard 10 in. square, how many square inches must necessarily be wasted? ($\pi = 3{\cdot}1416$).

Exp. 300. Illustrate, by drawing figures and by actual measurement, each of the following truths and its converse:—

Equal chords of a circle are equidistant from the centre, and the greater chord of a circle is nearer the centre than the less.

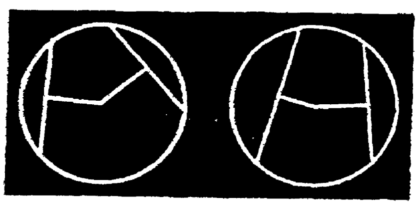


FIG. 188. FIG. 189.

Learn these by heart.

Exp. 301. In a circle of radius 2 cm. place a regular hexagon and calculate to the tenth of a millimetre the distance of each side of the hexagon from the centre of the circle. Verify by measurement.

Exp. 302. In a circle of radius 0·8 in. place chords of lengths 1·6 in., 0·6 in. and 1·1 in. respectively. Calculate to the hundredth of an inch and compare their distances from the centre. Verify by measurement.

TANGENCY.

Exp. 303. Draw a perpendicular AD from a point A to a straight line BC. Describe a circle with centre A and radius AD. Notice

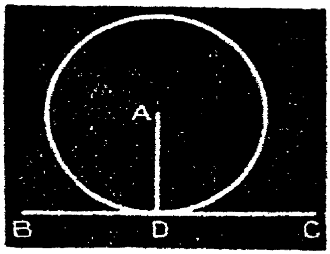


FIG. 190.

that this circle meets the line BC but does not cut it. BC is said to touch the circle and is called a tangent to it.

From this experiment we are led to conclude :—

> *The tangent at any point of a circle and the radius through the point are perpendicular to one another.*

Learn this by heart.

Exp. 304. Describe a circle of radius 1·9 cm. Take a point A distant 2·3 cm. from its centre. Calculate to the hundredth of a cm. the length of a tangent from A to the circle, and hence draw it. How many tangents can be drawn from A to the circle? Compare their lengths. Account for the result.

Exp. 305. Draw two concentric circles and draw several chords of the outer circle which touch the inner circle. Measure and compare their lengths. Account for the result. Calculate their lengths when the radii of the circles are 2·5 cm. and 0·7 cm. respectively. Verify by drawing and measurement.

Exp. 306. Describe a circle and at a point P on the circumference draw a chord PR and a tangent PQ so that $\angle$ RPQ = 50°. What fraction of the whole circumference is the smaller arc formed by the chord?

Exp. 307. Draw any angle ABC and its bisector BD. Take any point E on BD and draw EF, EG perpendicular to BA, BC respectively. Why is EF = EG? With centre E describe a circle to touch BA, BC. Describe two other circles to touch BA, BC.

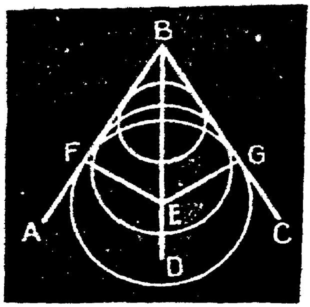


FIG. 191.

Exp. 308. Draw any triangle ABC. Draw the bisectors of its angles B and C. Let them meet at O. Why is O equidistant from

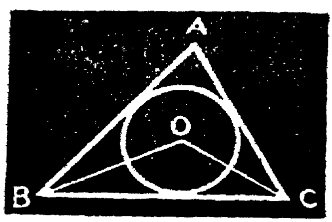


FIG. 192.

AB, BC, CA? With centre O describe a circle touching AB, BC, CA. Call it the in-circle of the triangle.

The radius of this in-circle is most conveniently found by drawing a perpendicular from O to any side of the triangle.

Exp. 309. Draw any triangle ABC. Describe a circle touching the side BC, the side AB produced and the side AC produced. Call it an e-circle of the triangle. Describe the other e-circles of the triangle ABC.

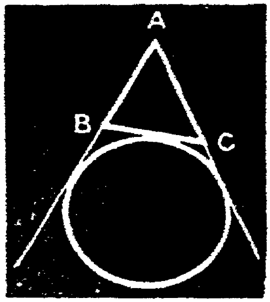


FIG. 193.

Exp. 310. Draw a straight line AB cutting another straight line CD at right angles. Describe several circles to touch CD, but having their centres in AB. Notice that these circles meet one

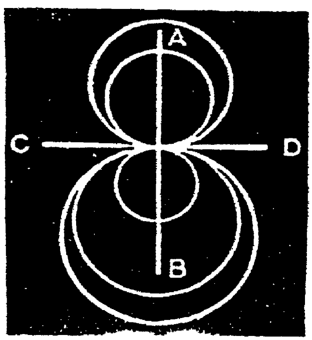


FIG. 194.

another at the same point, but do not cut one another. They are said to touch—some internally and some externally.

From this experiment we are led to conclude:—

If two circles touch, the point of contact lies on their line of centres.

Learn this by heart.

Exp. 311. If two circles touch, the distance between their centres is equal to the sum or difference of their radii. Account for this. Distinguish between the two cases, and illustrate by drawing and measurement when the radii are 1·1 in. and 0·7 in. respectively.

Exp. 312. Show by a dotted line in a carefully drawn figure the locus of the centre of a circle rolling (1) inside, (2) outside a fixed circle of greater radius.

Exp. 313. Describe three circles of radii 0·6 in., 0·7 in. and 0·8 in. respectively, so that each touches the other two externally.

Begin by drawing a rough diagram and ascertaining the sides of the triangle formed by joining the centres of the circles.

Exp. 314. Describe three circles of radii 2·8 cm., 1·9 cm. and 0·6 cm. respectively, so that two touch each other externally and the third internally.

Exp. 315. Describe two circles having the distance between their centres:—

(i) Greater than the sum of their radii.
(ii) Equal to the sum of their radii.
(iii) Less than the sum of their radii.
(iv) Equal to the difference of their radii.
(v) Less than the difference of their radii.

ANGLE PROPERTIES OF CIRCLES.

Exp. 316. Illustrate by drawing figures and by measurement with protractor the following truth:—

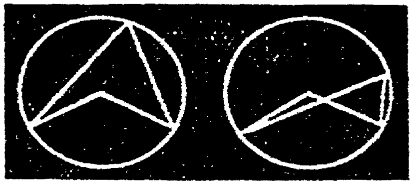


Fig. 195. Fig. 196.

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

Learn this by heart.

Exp. 317. Construct an isosceles triangle having each of the base angles 75°. Calculate the angle subtended by the base of the triangle at the centre of its circum-circle. Verify by drawing and measurement.

Exp. 318. Construct a triangle ABC from the following data: BC = 1·2 in., angle ABC = 50°, angle ACB = 79°. If O is the centre of its circum-circle, calculate angle ABO and angle ACO. Verify by drawing and measurement.

Exp. 319. ABCDEF is a regular hexagon. Calculate angle CFD. Verify by drawing and measurement.

Draw the hexagon inside a circle as in Exp. 279.

Exp. 320. A, B, C are points on the circumference of a circle whose centre is O such that angle ABC = 110° Calculate angle AOC. Verify by drawing and measurement.

Exp. 321. Any part of a circle bounded by an arc and its chord

Fig. 197.

is called a segment of the circle, and the angle subtended by the chord of a segment at any point on the arc is called an angle in the

segment. Illustrate by drawing figures and by measurement with protractor the following truth :—

Angles in the same segment of a circle are equal.

Learn this by heart.

Exp. 322. Draw an acute angle on thin cardboard and cut it out as in Exp 20. Mark two points A and B on a sheet of paper at a convenient distance apart. Place the angle in several positions

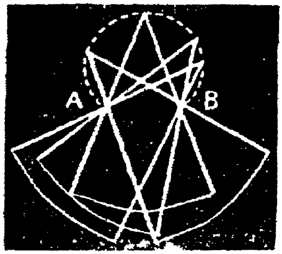


FIG. 198.

between the points, but let the two arms of the angle always pass one through A and the other through B. Trace the locus of the vertex of the angle. Do A and B lie upon the locus ? What is the locus ?

Exp. 323. Repeat Exp. 322 with (1) a right angle, (2) an obtuse angle.

From Exps. 322, 323 we are led to conclude :—

If the line joining two points subtends equal angles at two other points on the same side of it, the four points lie on the same circle.

Learn this by heart.

Exp. 324. Make any triangle ABC and draw AD, BE the perpendiculars from A and B upon BC and AC respectively. Describe

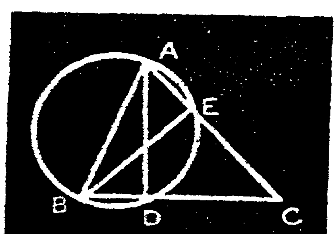


FIG. 199.

a circle to pass through A, B, D, E and explain why this is possible.

Exp. 325. On the same base AB and on the same side of it construct two triangles ABC, ABD such that angle ABC = 92°, angle BAC = 27°, angle ABD = 36°, angle BAD = 83°. Describe a circle to pass through A, B, C, D and explain why this is possible.

Exp. 326. A segment of a circle is called a major segment, a minor segment or a semi-circle according as its arc is greater than, less than or equal to half the circumference of the circle.

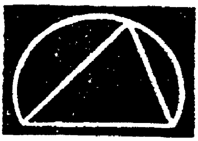


FIG. 200.

FIG. 201.

FIG. 202.

Illustrate by drawing figures and by measurement with protractor the following truths:—

The angle in a semi-circle is a right angle.
The angle in a major segment is acute.
The angle in a minor segment is obtuse.

Learn these by heart.

Exp. 327. In any circle draw a diameter AB and a chord AC so that angle BAC = 33°. Calculate angle ABC and verify by measurement.

Exp. 328. Calculate to the hundredth of an inch the radius of a circle passing through (0, 0), (11, 0), (0, 7) on inch paper. Verify by plotting and measurement.

Exp. 329. A rectilineal figure is said to be inscribed in a circle when all its vertices lie on the circumference of the circle.

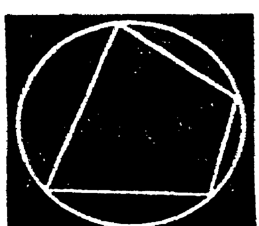


FIG. 203.

Illustrate by drawing figures and by measurement with protractor the following truth :—

The opposite angles of any quadrilateral inscribed in a circle are supplementary.

Learn this by heart.

Exp. 330. Make a triangle ABC from the following data : AB = 3·2 cm., angle A = 72°, angle B = 39°. Calculate the angle in the minor segment of the circum-circle of the triangle ABC formed by the chord AB. Verify by drawing and measurement.

Exp. 331. Draw a quadrilateral ABCD having angle A + angle C = 180°, and show that the circle passing through A, B and C will also pass through D.

From this experiment we are led to conclude :—

If a pair of opposite angles of a quadrilateral are supplementary its vertices lie on the same circle.

Learn this by heart.

Exp. 332. On the same base AB and on opposite sides of it construct two triangles ABC, ABD such that angle ABC = 73°, angle BAC = 24°, angle ABD = 62°, angle BAD = 21°. Describe a circle to pass through A, B, C, D, and explain why this is possible.

Exp. 333. At any point A on the circumference of a circle draw a tangent BAC and through A draw a chord AD making with BAC the angle DAC on the *right* of AD. Take any point E on the arc of the segment AED on the *left* of AD (called the "alternate

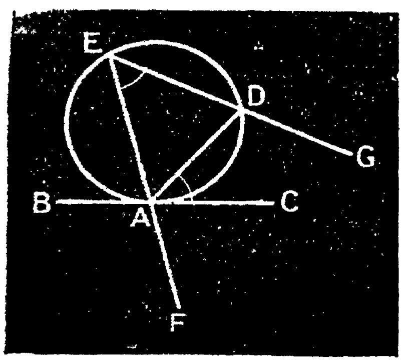


Fig. 204.

segment). Join EA, ED and produce EA, ED to F, G respectively. Now make a tracing of the angle FEG and move the tracing over the figure in such a way that EF always passes through

A and EG through D. Then E will move along the arc of the segment AED. (Why?) When E moves into coincidence with A, notice that EF lies along BAC and the angle FEG coincides with the angle CAD.

Exp. 334. Show as in Exp. 333 that the angle BAD is equal to the angle in the alternate segment.

From Exps. 333, 334 we are led to conclude :—

If a straight line touches a circle and from the point of contact a chord is drawn, the angles which this chord makes with the tangent are respectively equal to the angles in the alternate segments.

Learn this by heart.

Exp. 335. In Fig. 204 if $\angle$ CAD = 39° calculate the angle subtended by the chord AD at the centre of the circle. Verify by drawing and measurement.

Exp. 336. Describe a circle of radius 0·9 in. Draw a tangent to it at any point P on its circumference. From P draw a chord PQ to cut off a segment PRQ containing an angle of 63°.

Exp. 337. Describe a circle of radius 3·5 cm. Draw a chord to cut off a segment containing an angle of 103°.

Exp. 338. Draw a line AB measuring 1·1 in. At A in AB make an angle BAC of 42°. Describe a circle passing through A and B (its centre must lie on the perpendicular bisector of AB. Why?)

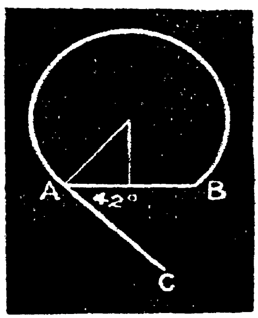


FIG. 205.

and also touching AC at A (its centre must lie on the perpendicular to AC at A. Why?). The segment on AB will contain an angle of 42°. Why?

Exp. 339. On a straight line measuring 3·1 cm. construct a segment containing an angle of 64°.

Exp. 340. Make triangles from the following data:—

(i) $BC = 2{\cdot}7$ cm $\angle BAC = 51°$ Perpendicular from A on $BC = 2{\cdot}5$ cm

(ii) $BC = 1$ in. $\angle BAC = 43°$. Area $= 0{\cdot}6$ sq. in.

Exp. 341 Describe a circle of radius 1 1 in. and from any point **P** on its circumference draw chords PQ, PR so that $\angle PQR = 29°$ and $\angle PRQ = 76°$.

Exp. 342. In a circle of radius 2·4 cm. inscribe a triangle equiangular to a triangle whose angles measure 115°, 27° and 38° respy.

CYLINDERS, CONES AND SPHERES

Figs. 206, 207 represent a solid which we shall call a circular cylinder. The ends of a circular cylinder are equal circles lying in

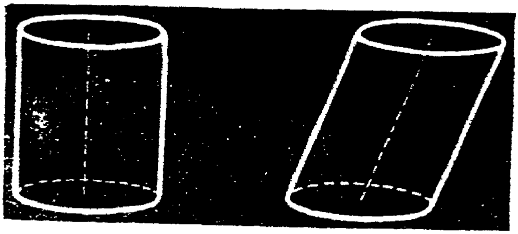


FIG. 206. FIG. 207.

parallel planes and the straight line joining the centres of these circles is called the axis of the cylinder.

A cylinder is said to be right if its axis stands upright as in Fig. 206 and skew or oblique if it leans over to one side as in Fig. 207.

Exp. 343. Cut out several rectangular strips of paper, all of the same width—say $1\frac{1}{2}$ in. Wrap them round and round your pencil and secure the end of the last strip with gummed paper. Now slip the roll off the pencil and cover the open ends with thin cardboard cut to size and fastened to the curved surface with gummed paper. You will thus have made a model of a right circular cylinder.

Exp. 344. Imagine a sharp knife to cut straight through your model of a right circular cylinder (1) along its axis, (2) along a straight line parallel to its axis, (3) along a straight line perpendicular to its axis, (4) along a straight line inclined to its axis, and make a sketch of the plane section that you believe to be obtained in each case.

Exp. 345. Cut out a paper rectangle and fasten a piece of string with gummed paper along one of its sides. Now twirl the rectangle by means of the string and notice that it "generates" a right circular cylinder whose axis lies along the line of the string.

The form of the generated cylinder can be seen more clearly by fastening the string along the line midway between two opposite sides of a rectangle. Now, by means of the string, we can make two rectangles, equal in all respects, revolve about a common side, and so double the effect.

Figs. 208, 209 represent a solid which we shall call a circular cone.

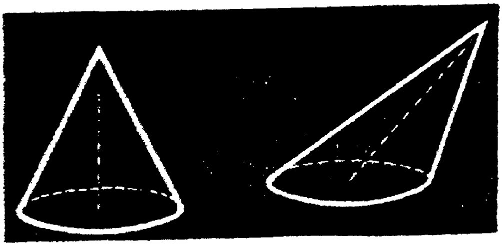


FIG. 208. FIG. 209.

The ends of a circular cone are a point at which it tapers away called its apex, and a circle called its base, and the straight line joining the apex to the centre of the base is called the axis of the cone.

A cone is said to be right if its axis stands upright as in Fig. 208 and skew or oblique if it leans over to one side as in Fig. 209.

Exp. 346. Cut out of thin cardboard a sector of a circle of radius about $2\frac{1}{2}$ in., and let the angle between the bounding radii of the sector be about 160°. Bend the sector round until its bounding radii lie along one another and join them with gummed paper. You will thus have made a model showing the curved surface of a right circular cone. For base, cut out a circle of thin cardboard to size and fasten it to the curved surface with gummed paper.

Exp. 347. Imagine a sharp knife to cut straight through your model of a right circular cone (1) along its axis, (2) along a straight line parallel to its axis, (3) along a straight line perpendicular to its axis, (4) along a straight line inclined to its axis, and make a sketch of the plane section that you believe to be obtained in each case.

Exp. 348. Cut out a paper right-angled triangle and fasten a piece of string with gummed paper along one of its sides containing the right angle. Now twirl the triangle by means of the string, and notice that it generates a right circular cone whose axis lies along the line of the string.

The form of the generated cone can be seen more clearly by fastening the string along the median of an isosceles triangle passing through the vertex. Now, by means of the string, we can make two right-angled triangles, equal in all respects, revolve about a common side, and so double the effect.

Fig. 210 represents a solid which we shall call a sphere. All points on the surface of a sphere are equidistant from a certain point within the solid called the centre of the sphere.

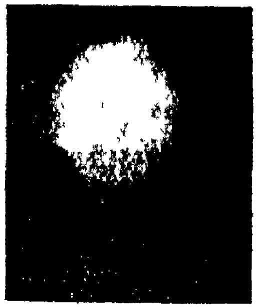


FIG. 210.

Exp. 349. Imagine a sharp knife to cut straight through a sphere (1) passing through its centre, (2) not passing through its centre, and make a sketch of the plane section that you believe to be obtained in each case.

Exp. 350. Cut out a paper semi-circle and fasten a piece of string with gummed paper along its diameter. Now twirl the semi-circle by means of the string, and notice that it generates a sphere whose centre lies upon the line of the string.

The form of the generated sphere can be seen more clearly by fastening the string along a diameter of a circle. Now, by means of the string, we can make two equal semi-circles revolve about a common diameter, and so double the effect.

The cylinder, cone and sphere are called solids of revolution, because each can be generated by the revolution of a plane figure about an axis.

THEORETICAL SECTION.

INTRODUCTORY.

A unit of length or linear unit is a line of definite length chosen for the measurement of other lines. If a square is described upon a linear unit, a surface of definite area is obtained, since squares described on equal straight lines are equal in area—easily proved by superposition. By means of this surface of definite area, other surfaces can be measured, and it is called the **unit of area** or **square unit** that "corresponds" to the linear unit from which it has been derived. Thus a square of side 1 in., called a square inch, is the square unit that corresponds to the linear unit called an inch.

Figures of equal area are said to be **equivalent**, and the symbol "=" is used to denote the equivalence of two figures.

AREAS OF RECTANGLES.

Theorem 26.

Gen. Enun. *The number of any linear unit in the base of a rectangle multiplied by the number of the same linear unit in the altitude gives the number of the corresponding square unit in the area.*

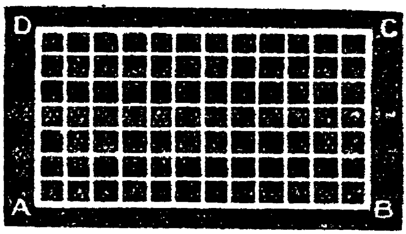


Fig. 211.

Part. Enun. Let ABCD be a rect. whose base AB measures a of any linear unit and whose altitude BC measures b of the same linear unit where a and b are whole numbers.

It is reqd. to prove that

rect. ABCD *contains* $a \times b$ *of the corresponding sq. unit.*

Const. Suppose AB divided into a equal parts and BC into b equal parts so that each of these parts will be the linear unit.

Through the pts. of division in BC suppose st. lines drawn || to AB.

Through the pts. of division in AB suppose st. lines drawn || to BC.

Proof. The rect. is now divided into b rows, each row containing a squares and each square being the sq. unit that corresponds to the linear units into which AB and BC have been divided.

∴ the rect. ABCD contains $a \times b$ of the corresponding sq. unit.

Q. E. D.

Note.—In the proof of Theorem 26 we assume that both the base and the altitude of the rectangle contain the chosen linear unit an exact number of times, in other words, that a and b are *whole numbers*. If a or b are *fractional*, that is to say if the chosen linear unit is not con-

tained an exact number of times in the base and altitude of the rectangle, we can always find a smaller linear unit that is a common measure of these two lengths since they are commensurable, and then the rule of Theorem 26 can be seen to apply.

For example:—

$$\begin{aligned}\text{Area of rectangle } 3\tfrac{1}{3}\text{ ft. by } 4\tfrac{3}{4}\text{ ft.} &= \text{area of rectangle 40 in. by 57 in.}\\ &= 40 \times 57 \text{ sq. in.}\\ &= \frac{40 \times 57}{144} \text{ sq. ft.}\\ &= 3\tfrac{1}{3} \times 4\tfrac{3}{4} \text{ sq. ft.}\end{aligned}$$

Or again,

$$\begin{aligned}\text{Area of rectangle 4·3 cm. by 3·7 cm.} &= \text{area of rectangle 43 mm. by 37 mm.}\\ &= 43 \times 37 \text{ sq. mm.}\\ &= \frac{43 \times 37}{100} \text{ sq. cm.}\\ &= 4{\cdot}3 \times 3{\cdot}7 \text{ sq. cm.}\end{aligned}$$

When the base and altitude of a rectangle are *incommensurable* we can, for practical purposes, substitute another rectangle whose base and altitude are commensurable and whose area differs from the area of the given rectangle by a quantity less than any that can be assigned. For we can choose a linear unit *as small as we please* and describe a rectangle whose base and altitude contain this unit exactly, and differ from the base and altitude of the original rectangle by less than any assignable quantities. Or, what is the same thing, we can express the base and altitude of the original rectangle as decimal fractions of any linear unit to as many places as we please, and then take these approximate lengths for their true lengths.

The circumference and diameter of a circle are examples of incommensurable lengths, and so are a side and diagonal of a square. As a matter of fact, in any figure it is the rule rather than the exception to find lines that are incommensurable.

Cor. 1. *The square of the number of any linear unit in the side of a square gives the number of the corresponding square unit in the area.*

Cor. 2. *Rectangles on equal bases and of the same altitude are equivalent.* (Proof by Superposition.)

Cor. 3. *Equivalent rectangles on equal bases are of the same altitude.* (Proof by Reductio ad Absurdum.)

Cor. 4. *Equivalent rectangles of the same altitude stand on equal bases.*

Two adjacent sides of a rectangle completely determine its shape and size, hence a rectangle is said to be *contained* by any two adjacent sides.

Thus the rectangle ABCD is said to be contained by AB, AD: and if EF, GH are two straight lines such that EF = AB and GH = AD, the rectangle ABCD may also be said to be contained

by EF, GH. The abbreviation "rect. AB, AD" or "AB × AD"

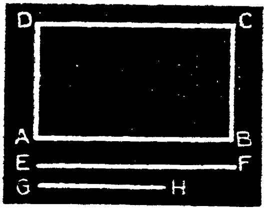


FIG. 212.

or "AB . AD" is used to denote "the rectangle contained by AB, AD".

Similarly a side of a square completely determines its shape and size, and the abbreviation "sq. on AB" or "AB^2" is used to denote "the square having AB for side".

Thus such an expression as

$$AB \,.\, BC + BC^2$$

denotes

> "The rectangle having AB and BC for adjacent sides together with the square having BC for side".

Squares and rectangles are often denoted by the two letters at opposite corners. Thus the rectangle in Fig. 212 can be described as rect. AC or rect. BD.

AREAS OF PARALLELOGRAMS.

THEOREM 27.

Gen. Enun. *A parallelogram is equivalent to the rectangle on the same base and between the same parallels.*

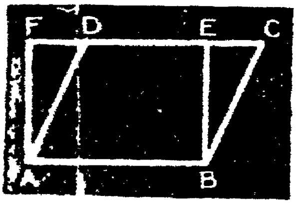


FIG. 213.

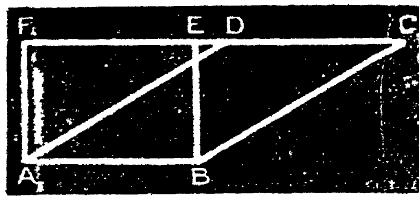


FIG 214

Part. Enun. Let ABCD, ABEF be a ▱gm and a rect. on the same base AB and between the same ||s AB, FC.

It is reqd. to prove that

▱*gm* ABCD = *rect.* ABEF.

Proof. In the △s BEC, AFD

∵ { ext. ∟ BEC = int. ∟ AFD . . . Th. 6.
int. ∟ BCE = ext. ∟ ADF . . . Th. 6.
BE = AF . . . . . Th. 22.

∴ △ BEC ≡ △ AFD . . . . . Th. 11.

Now from Fig. ABCF take away △ BEC
and rect. ABEF remains,
and from Fig. ABCF take away △ AFD
and ▱gm ABCD remains.

∴ ▱gm ABCD = rect. ABEF. Q.E.D.

Cor. 1. *A parallelogram is equivalent to the rectangle on the same base and of the same altitude.*

Hence,

The number of any linear unit in the base of a parallelogram multiplied by the number of the same linear unit in the altitude gives the number of the corresponding square unit in the area.

Cor. 2. *Parallelograms on the same base or on equal bases and of the same altitude are equivalent.*

Exercises.

602. *If a square and a rhombus stand on the same base, the square has the greater area.*

603. ABCD is a ▱gm and E, F are the mid-pts. of AB, CD; prove that EBFD is a ▱gm. = △ ACD.

604. ABCD is a quadl. whose diag. AC bisects its other diag. BD; if the ▱gms ACBE, ACDF are completed, prove that they are equiv.

605. *Deduce a rule for calculating the area of a △ from the theorem. ▱gms on the same base and between the same ||s are equiv.* (Mad. Mat.)

606. *Equiv. ▱gms between the same ||s are on = bases.*

607. *Equiv. ▱gms on the same base and on the same side of it are between the same ||s.*

* 608. ABC is a △ rt. ∟d at A and squares BCDE, CFGA, AHKB are descd. all externally; if the ▱gms CFLD, BKME are completed, prove that they are equiv.

* 609. ▱gms AFGC, CBHK are descd. on the sides AC, BC outside the △ ABC, FG, KH are produced to meet in L; LC is joined, and through A and B, AD, BE are drawn || to LC, meeting LF, LK in D, E; prove that ABED is a ▱gm = ▱gm AFGC + ▱gm CBHK. (Mad. Mat.)

AREAS OF TRIANGLES.

THEOREM 28.

Gen. Enun. *A triangle is equivalent to half the rectangle on the same base and between the same parallels.*

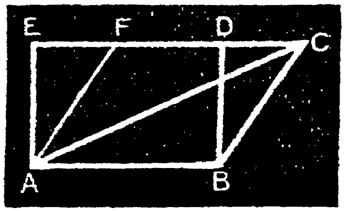


FIG. 215.

Part. Enun. Let ABC, ABDE be a △ and a rect. on the same base AB and between the same ||s AB, EC.

It is reqd. to prove that

△ ABC = ½ *rect.* ABDE.

Const. Suppose AF to have been drawn || to BC meeting DE (or DE produced) in F. Then ABCF is a ▱gm.

Proof. ∵ △ ABC = ½ ▱gm ABCF . . .	Th. 22.
and ▱gm ABCF = rect. ABDE	Th. 27.
∴ △ ABC = ½ rect. ABDE.	Q.E.D.

Cor. 1. *A triangle is equivalent to half the rectangle on the same base and of the same altitude.*

Hence,

Half the number of any linear unit in the base of a triangle multiplied by the number of the same linear unit in the altitude gives the number of the corresponding square unit in the area.

Cor. 2. *Triangles on the same base or on equal bases and of the same altitude are equivalent.*

Cor. 3. *The area of a trapezium whose parallel sides and height measure a, b and h linear units respectively is $\frac{1}{2}(a + b)h$ square units.*

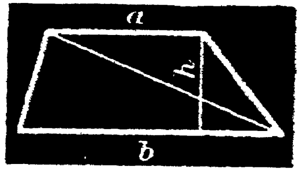


FIG. 216.

Proof. Draw a diag., then

$$\begin{aligned}\text{Area of trapezium} &= \text{sum of areas of 2 } \triangle\text{s.}\\ &= (\tfrac{1}{2}ah + \tfrac{1}{2}bh) \text{ sq. units.}\\ &= \tfrac{1}{2}(a + b)h \text{ sq. units.}\end{aligned}$$

Q.E.D.

Exercises.

610. *A $\triangle$ is bisected by each of its medians.*

611. S is any pt. in the median PT of a $\triangle$ PQR; prove that $\triangle$ PQS = $\triangle$ PRS.

612. ABC is a $\triangle$ and DE is ‖ to BC, cutting AB, AC in D, E; prove that $\triangle$ ABE = $\triangle$ ACD.

613. ABCD is a trapezium having AB ‖ to DC, and E is the intersection of its diags.; prove that $\triangle$ AED = $\triangle$ BEC.

614. ABCD is a quadl. whose diag. AC bisects the other diag. BD; prove that $\triangle$ ABC = $\frac{1}{2}$ quadl. ABCD.

615. E is a pt. on the side AD of a $\square$gm ABCD; prove that $\triangle$ EAC + $\triangle$ EBD = $\frac{1}{2}$ $\square$gm ABCD. (Mad. Mat.)

616. AB is divided at C, D, E so that AC = CD = DE = EB and F is an outside pt.; compare $\triangle$ AFE with $\triangle$ AFB.

617. The base and height of one $\triangle$ are respy. $\frac{1}{3}$ and $\frac{1}{2}$ the base and height of another; compare their areas.

618. *Equivalent $\triangle$s between the same ‖s have = bases.* (Punj. F. E.)

619. ABCD is a trapezium whose diags. intersect at E; if AE = EC then BE = ED.

620. ABCD is a $\square$gm whose diags. intersect at E; prove that $\triangle$ EAB = $\frac{1}{4}$ $\square$gm ABCD.

621. *The area of a rhombus = $\frac{1}{2}$ rect. contd by its diags.*

622. G *is the centroid of a* $\triangle$ PQR; *prove that* $\triangle$ PGR = $\frac{1}{3}$ $\triangle$ PQR.

623. PQR is a $\triangle$ whose medians QS, RT intersect at G; prove that $\triangle$ QRG = quadl. PTGS.

624. *A trapezium is bisected by the line joining the mid. pts. of its ‖ sides.*

625. ABC, DBC are $\triangle$s on the same base and between the same ‖s, and ABCK is a $\square$gm, and AC, BD meet in O; prove that the difference between $\triangle$s BOC, AOD = $\triangle$ DCK.

626. *When 2 sides of a △ are given the area is a maximum when their contained $\angle$ is a rt. $\angle$.* (Mad. Mat.)

627. *If the mid. pts. of any 2 sides of a △ are joined the △ so cut off is $\frac{1}{4}$ of the whole.* (Punj. Inter.)

628. The sides AB, AC of a △ ABC are produced to D, E so that BD = AB and CE = $\frac{1}{2}$ AC; prove that △ ADE = 3 △ ABC.

629. Through A, B, C, the angular pts. of a △, 3 ‖ lines are drawn to meet the opp. sides of the △ (produced if necessary) in D, E, F; prove that △ DEF = 2 △ ABC. (Mad. Mat.)

630. ABCD is a rect. having AB = 2 BC; E, F are pts. in BC, CD so that BE = EC and CF = 3 FD; prove that △ AEF = $\frac{7}{16}$ rect. ABCD.

631. AD is a median of the △ ABC; CE is drawn ‖ to DA, meeting BA produced in E; prove that △ AEC = 2 △ ABD.

632. D, E are pts. on the sides AB, AC of a △ ABC such that AD = $\frac{2}{3}$ AB and AE = $\frac{3}{4}$ AC; prove that △ ADE = $\frac{1}{2}$ △ ABC.

633. In the △ ABC, AB is produced beyond B to P, BC beyond C to Q, and CA beyond A to R, so that BP, CQ, AR are double AB, BC, CA respy.; prove that △ PQR = 19 △ ABC.

634. *Prove the following construction for bisecting a △* ABC *by a line drawn from any pt.* D *in the side* BC: *Join* E, *the mid. pt. of* BC, *to* A; *draw* EF ‖ *to* AD; *join* DF.

635. Through the mid. pt. of the side AB of a △ ABC a st. line is drawn cutting CA, CB in D, E; a ‖ st. line through C meets AB in F; prove that △ ADF = △ BEF.

* 636. *If 2 △s have 2 sides of the one equal to 2 sides of the other, each to each, and the contained $\angle$s supplementary, the △s are equiv.* (Mad. Mat.)

* 637. *The sum of the distances of any pt. within a regular polygon from the sides is constant for all positions of the pt.*

* 638. D, E are the mid. pts. of the sides AB, AC of a △ ABC and F, G are the pts. of trisection of the base BC; DF, EG are produced to meet at H; prove that △ FGH = $\frac{1}{3}$ △ ABC.

* 639. ABC is a △; D, E are the mid. pts. of AB, AC; BE, CD meet at O; prove that the area of the △ whose sides are equal in length to AO, BO, CO is $\frac{1}{3}$ △ ABC.

* 640. *The st. line joining the mid. pts. of the ‖ sides of a trapezium passes through the intersection of its diags.* (Bom. Sch. Fin.)

* 641. D, E are the pts of trisection of the base BC of a △ ABC and DF, EG are drawn ‖s to BA, CA, cutting one another at H and the sides of the △ at F, G; prove that fig. AFHG + △ DHE = $\frac{1}{3}$ △ ABC.

* 642. PR is a diag. of a sq. PQRS and T, X are the mid. pts. of the sides PS, SR respy.; if QT, QX meet PR in V, Z, prove that sq. PQRS = 3 fig. TSXZV.

* 643. PR is a diag. of a sq. PQRS and T is a pt. on PR such that PT = $\frac{1}{4}$ PR; if ST produced meets PQ in X, prove that △ TPX = $\frac{1}{3}$ △ TPQ.

THEOREM 29.

Gen. Enun. *Equivalent triangles on the same base are of the same altitude.*

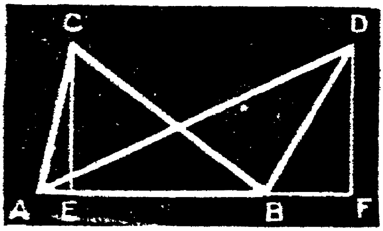


FIG. 217.

Part. Enun. Let ABC, ABD be 2 equiv. $\triangle$s on the same base AB, and let CE, DF be their altitudes respy.

It is reqd. to prove that

CE = DF.

Proof. $\because$ { $\triangle$ ABC = $\frac{1}{2}$ rect AB, CE . . Th. 28, Cor. 1.
{ $\triangle$ ABD = $\frac{1}{2}$ rect AB, DF . . Th. 28, Cor. 1.
and $\triangle$ ABC = $\triangle$ ABD. . . . Hyp.
$\therefore$ rect. AB, CE = rect. AB, DF.
$\therefore$ CE = DF . . . . Th. 26, Cor. 3.

Q.E.D.

Cor. *Equivalent triangles on equal bases are of the same altitude.*

Exercises.

644. *The st. lines bisecting the* $\lfloor$s *at the base of an isos.* $\triangle$ *meet the sides in* D *and* E. *Show that* DE *is* $\parallel$ *to the base.* (Punj. Mat.)

645. *The diag.* AC *of a* $\square$*gm* ABCD *is equidistant from* B *and* D.

646. In AC, a diag. of the $\square$gm ABCD, any pt. X is taken, and XB, XD are drawn. Show that $\triangle$ BAX = $\triangle$ DAX. (Mad. F. E.)

647. *Equivalent* $\triangle$s *are upon the same base and on the same side of it. Find the locus of their vertices.* (Cal. Mat.)

648. Q, R, S are pts. on the same st. line. Find the locus of a fourth pt. P which moves so that $\triangle$ PQR + $\triangle$ PRS = a constant.

649. QRST is a quadl. Find the locus of a pt. P which moves so that the pentagon PQRST is of constant area.

650. *Of all* $\triangle$s *on the same base and of the same area the isos.* $\triangle$ *has the least perimeter.* (Cal. Mat.)

651. *The st. line joining the mid. pts. of the sides of a △ is ‖ to the base.*

652. *If 2 equivalent △s stand on opp. sides of the same base, the st. line joining their vertices is bisected by their base or base produced.* (Bom. Mat.)

653. *If a quadl. is bisected by each of its diags. it is a ▱gm.* (Allah. Mat.)

654. *If the diags. of a quadl. divide it into 4 = parts, it is a ▱gm.*

* 655. ABCD *is a trapezium, and* E, F *are the mid. pts. of the non-‖ sides* BC, DA ; *prove that* EF *is ‖ to* AB.

* 656. APB, ADQ are 2 st. lines such that △ PAQ = △ BAD ; if the ▱gm ABCD is completed and BQ joined cutting CD in R, show that CR = AP. (Bom. Mat.)

THEOREM 30.

(Euc. I. 41.)

Gen. Enun. *A triangle is equivalent to half a parallelogram on the same base and between the same parallels.*

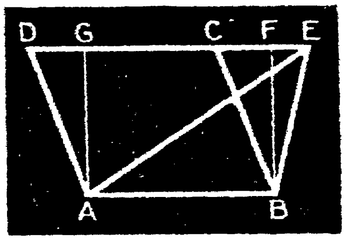


FIG. 218.

Part. Enun. Let ABE be a △ and ABCD a ▱gm on the same base AB and between the same ||s AB, DE.

It is reqd. to prove that

△ ABE = $\frac{1}{2}$ ▱*gm* ABCD.

Const. Suppose the rect. ABFG to have been descd. upon the base AB and between the ||s AB, DE.

Proof. ∵ △ ABE = $\frac{1}{2}$ rect. ABFG . . . . . Th. 28.
and rect. ABFG = ▱gm ABCD . . . . . Th. 27.
∴ △ ABE = $\frac{1}{2}$ ▱gm ABCD . . . . Q.E.D.

Exercises.

657. If 2 = st. lines cut one another at rt. ∟s, the quadl. formed by joining their extremities = $\frac{1}{2}$ the sq. on either st. line. (Bom. Mat.)

658. The area of any ▱gm is double the area of the quadl. formed by joining the mid. pts. of its adj. sides.

659. The area of any ▱gm ABCD is 4 times the area of the △ formed by joining A and B to any pt. P midway between AB and CD.

660. P is a pt. within a ▱gm ABCD. Show that △ APD + △ BPC = $\frac{1}{2}$ ▱gm ABCD. (Bom. Sch. Fin.)

661. P is a pt. within a ▱gm ABCD. Show that the difference of the △s APD and APB = △ APC. (Bom. Prev.)

662. E, F are pts. in the sides CD, DA of a ▱gm ABCD. Prove that △ ABE = △ BCF.

663. *The area of any quadl. is double the area of the quadl. formed by joining the mid. pts. of its adj. sides.* (Bom. Sch. Fin.)

664. ABCD *is a* ▱*gm; lines* EF, GH *are drawn* || *to* AD, AB, *the former cutting* AB, CD *in* E, F, *and the latter cutting* AD, BC *in* G, H. *Prove that* ▱gm DH + ▱gm HE = 2 △ AFH.

665. E is the intersection of the diags. of a ▱gm ABCD and F is any pt. in AD; prove that fig. BFCE = ½ ▱ gm. ABCD.

666. ABCD is a rect.; E is any pt. in BC and F in CD; show that 2 × △ AEF + rect. BE, DF = rect. ABCD (Bom. Mat.)

667. ABCD is a ▱gm, P any pt. in AC. If GPH, KPL are drawn || to the sides AD, AB respy., prove that the difference between the ▱gms PHCL, PGAK = 2 △ PBD. (Mad. F. A.)

668. Through D, E, the mid. pts. of the sides BC, CA of the △ ABC, any two || st. lines are drawn meeting AB in F, G; prove that DEGF is a ▱gm = ½ △ ABC. (Cal. F. E.)

* 669. ABCD *is a quadl. having* BC || *to* AD; E *is the mid. pt. of* DC; *show that* △ AEB = ½ *the quadl.* (Cal. F. E.)

* 670. AD is a median of the △ ABC and BE, CF are ||s to AD meeting any st. line through A at E, F; prove that fig. EBCF = 2 △ ABC.

* 671. *The area of a quadl = the area of a △ having 2 sides equal to the diags. of the quadl. and the contained* ∟ = *that between the diags.* (Cal. Mat.)

* 672. ABCD is a ▱gm and DF is drawn cutting AB in E and CB produced in F; prove that △ ABF = △ CFE.

* 673. E is the intersection of the diags. of a ▱gm ABCD and F is any pt. inside △ AEB; prove that the sum of △s AFC, BFD = the difference of △s AFB, CFD.

Def. 47. The parallels to the sides of a parallelogram through any point in one of the diagonals divide it into four parallelograms of which the two through which the diagonal passes are called the

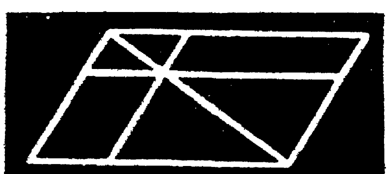


FIG. 219.

parallelograms about a diagonal and the other two are called the complements of the parallelograms about a diagonal.

THEOREM 31.

(Euc. I. 43.)

Gen. Enun. *The complements of the parallelograms about a diagonal of any parallelogram are equivalent.*

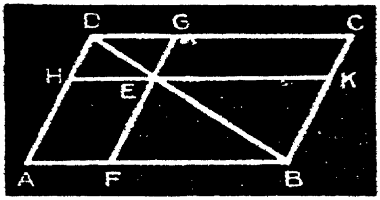


FIG. 220.

Part. Enun. Let ABCD be a ▱gm, and let HF, GK be the complements of the ▱gms FK, HG about the diagonal DB.

It is reqd. to prove that

complement HF = *complement* GK.

Proof. ∵ the diag. DB bisects the ▱gm ABCD Th. 22.

∴ the whole △ ABD = the whole △ CBD.

Similarly the parts, △s FBE, HED together = the parts, △s KBE, GED.

∴ the remainder, the complement HF = the remainder, the complement GK. Q.E.D.

Exercises.

674. *In the fig. of Th. 31 prove that* ▱*gm* HC = ▱*gm* AG.

675. *Prove that each of the* ▱*gms about the diags. of a rhombus is a rhombus.*

676. *Through a pt.* E *within a* ▱*gm* ABCD *lines are drawn* ‖ *to the sides making* ▱*gm* AE = ▱*gm* EC ; *prove that* E *lies on the diag.* BD. (Bom. Prev.)

677. In the fig. of Th. 31 a pt. T is taken on BD produced : prove that △ TAH = △ TGC.

* 678. E is a pt. within a ▱gm ABCD ; prove that the distance from C to the st. line joining A to E = the difference of the distances from B and D.

* 679. Through a pt. E within a ▱gm ABCD lines are drawn ‖ to the sides; prove that △ AEC = ½ difference between ▱gms DE, BE. (Mad. F. A.)

* 680. What is the greatest value which the complements of the ▱gms about the diag. of any given ▱gm can have ?

THEOREM 32.

(Theorem of Pythagoras.)

Gen. Enun. *In a right-angled triangle the square on the hypotenuse is equivalent to the sum of the squares on the sides containing the right angle.*

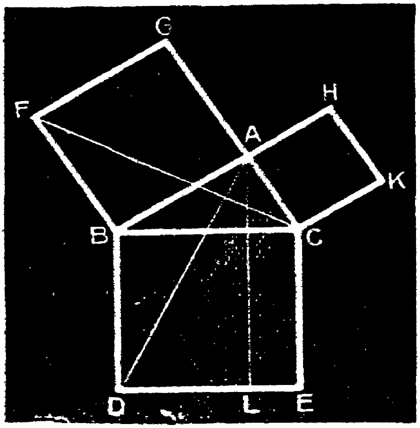


FIG. 221.

Part. Enun. Let ABC be a right-angled △ having ∟ A a rt. ∟.
Let BE, CH, AF be squares descd. upon BC, CA, AB respy.
It is reqd. to prove that

$$BC^2 = CA^2 + AB^2.$$

Const. Join CF, AD.
Through A suppose AL drawn ∥ to BD meeting DE at L.

Proof. ∵ the adj. ∟s BAC, BAG are rt. ∟s
∴ CA, AG are in the same st. line . . . Th. 2.
Again ∵ ∟s FBA, CBD, are rt. ∟s
∴ ∟ FBA + ∟ ABC = ∟ CBD + ∟ ABC,
that is ∟ FBC = ∟ ABD.
Hence, in the △s FBC, ABD

$$\because \begin{cases} FB = AB. \\ BC = BD. \\ \angle FBC = \angle ABD. \end{cases}$$

∴ △ FBC ≡ △ ABD . . . . Th. 10.
But △ FBC = ½ sq. AF (on the same base and between the same ∥s) . . . . Th. 28.

And $\triangle$ ABD = $\frac{1}{2}$ rect. BL, for the same reason.
∴ rect. BL = sq. AF.
Similarly by joining BK and AE it can be proved that
rect CL = sq. CH.
∴ rect. CL + rect. BL = sq. CH + sq. AF,
that is, $BC^2 = CA^2 + AB^2$ Q.E.D.

Exercises.

681. *The area of a sq. = $\frac{1}{2}$ the area of the sq. on its diag.*

682. *The diag. of a sq. = side of sq. $\times \sqrt{2}$.*

683. PS is $\perp$ to the base of a $\triangle$ PQR; prove that $PQ^2 + RS^2 = PR^2 + QS^2$.

684. ABC is a $\triangle$ rt. $\angle$d at A and BE, CF are medians; prove that $4(BE^2 + CF^2) = 5BC^2$. (Cal. F. E.)

685. ABCD is a quadl. having AC $\perp$ to BD; prove that $AB^2 + CD^2 = BC^2 + DA^2$.

686. *The sum of the squares on the sides of a rhombus = the sum of the squares on the diags* (Cal Mat.)

687. The intersection of the diags. of the sq. descd. on the hypot. of a rt. $\angle$d $\triangle$ is equally distant from the sides containing the rt. $\angle$. (Cal. Mat.)

688. BD is a diag. of the sq. ABCD and BE bisects $\angle$ ABD meeting CD produced in E; prove that $DE^2 = 2 \times$ sq. ABCD.

689. If a st. line is drawn from an acute $\angle$ of a rt. $\angle$d $\triangle$ bisecting the opp. side, the sq. on that line = sq. on hypot. $- 3 \times$ sq. on half the line bisected. (Bom. Sch. Fin.)

690. *In a rt. $\angle$d $\triangle$ if one of the acute $\angle$s is double the other the sq. on the greater of the sides containing the rt. $\angle$ = 3 times the sq. on the lesser side.* (Bom. Sch. Fin.)

691. *The height of an equilat. $\triangle$ = side $\times \frac{\sqrt{3}}{2}$.*

692. *The area of an equilat. $\triangle$ = sq. on side $\times \frac{\sqrt{3}}{4}$.*

693. *In a rt. $\angle$d $\triangle$ the equilat. $\triangle$ on the hypot. = the sum of the equilat. $\triangle$s on the sides containing the rt. $\angle$.*

* 694. ABC is an equilat. $\triangle$ and AB is produced to D so that BD = 2AB; prove that $CD^2 = 7AB^2$. (Mad. Mat.)

* 695. Any rectangle = $\frac{1}{2}$ rect. contained by the diags. of the sqs. descd. on 2 of its adjacent sides. (Cal. F. E.)

* 696. ABC is a $\triangle$ rt. $\angle$d at B; prove that the rect. contained by the diags. of the sqs. descd. on AB and BC = 4 $\triangle$ ABC.

* 697. *If any pt.* P *is joined to the angular pts. of a rectangle* ABCD, $PA^2 + PC^2 = PB^2 + PD^2$. (Bom. Mat.)

* 698. D, E, F *are the feet of the $\perp$s from the angular pts. of a $\triangle$* ABC *to the opp. sides; prove that* $AE^2 + BF^2 + CD^2 = AF^2 + BD^2 + CE^2$. (Bom. Prev.)

* 699. Two $\triangle$s ABC, A′B′C′ have their sides respectively ∥. BB_1 and CC_1 are drawn ⊥s to B′C′; CC_2 and AA_2 to C′A′; AA_3 and BB_3 to A′B′. Prove that $AB_1^2 + BC_2^2 + CA_3^2 = AC_1^2 + BA_2^2 + CB_3^2$. (Camb. Tripos.)

* 700. PQR is a rt. ∟d $\triangle$ on a fixed hypot. PQ; PRS, QRT are equilat. $\triangle$s on PR, QR; X, V are the mid. pts. of PR, QR; prove that $\triangle$ PVT + $\triangle$ QXS = a constant.

* 701. If CE, BD are the sqs. descd. upon the side AC and the hypot. AB of a rt. ∟d $\triangle$ ABC, prove that BE is ⊥ to CD.

* 702. If CE, BD are the sqs. descd. upon the side AC and the hypot. AB of a rt. ∟d $\triangle$ ABC, and if BE, CD intersect in F, prove that AF bisects ∟ EFD. (Camb. Tripos.)

* 703. If AD, AG are the sqs. descd. upon the sides AC, AB containing the rt. ∟ of a rt. ∟d $\triangle$ ABC, and if BD, CG cut AC, AB in M, N respy., prove that AM = AN.

* 704. Which of Exercises 701, 702, 703 are true whether the $\triangle$ ABC is right-angled or not?

THEOREM 33.

(Euc. I. 48.)

Gen. Enun. *In a triangle, if the square on one side is equivalent to the sum of the squares on the other two sides, then the angle contained by these two sides is a right angle.*

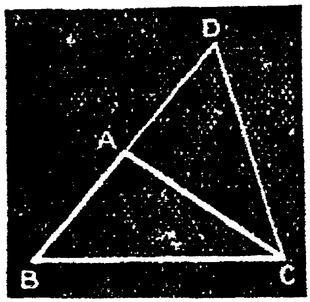


FIG. 222.

Part. Enun. Let ABC be a $\triangle$ such that $BC^2 = AB^2 + AC^2$.
It is reqd. to prove that

$\angle$ BAC *is a rt.* $\angle$.

Const. Suppose AD drawn $\perp$ to AC and = AB.
Join DC.

Proof. $\because \angle$ CAD is a rt. $\angle$ . . . . . . Const.

$\therefore DC^2 = AD^2 + AC^2$ . . . . . . Th. 32

$= AB^2 + AC^2$ . . . . . . Const.

$= BC^2$ . . . . Hyp.

$\therefore$ DC = BC.

Hence in $\triangle$s BAC, DAC

$$\because \left\{ \begin{array}{l} AB = AD. \\ AC \text{ is common.} \\ BC = DC. \end{array} \right.$$

$\therefore \angle BAC = \angle DAC$ . . . . . Th. 14

$= \text{a rt. } \angle$. Q.E.D.

Exercises.

705. ABCD is a quadl. rt. $\angle$d at B, and $AD^2 = AB^2 + BC^2 + CD^2$ prove that $\angle$ ACD is a rt. $\angle$. (Bom. Prev.)

706. P is a pt. outside a sq. ABCD such that the sum of the sqs. on the distances of P from the sides of the sq. $= AC^2$; prove that $\angle$ APC is a rt. $\angle$.

* 707. *In a △ if the sq. on one side > the sum of the sqs. on the other two sides, the ∠ contained by these sides is obtuse.*

* 708. *In a △ if the sq. on one side < the sum of the sqs. on the other two sides, the ∠ contained by these sides is acute.*

* 709. If the sides of a △ measure n, $n + 1$, $\sqrt{2n + 1}$ linear units respy., the △ is rt. ∠d.

* 710. If 2 sides of a △ measure $4n^2 - 1$ and $4n^2 + 1$ linear units respy., what must the third side measure in order that the △ may be rt. ∠d?

CIRCLES.

We have already defined a circle, its circumference, centre, radius and diameter (p. 38), but, for convenience, these definitions are repeated here.

Def. 13. A circle is a plane figure contained by one line which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another. This point is called the centre of the circle.

Circles which have the same centre are said to be concentric.

Def. 14. Any straight line drawn from the centre of a circle to its circumference is called a radius of the circle, and any straight line drawn through the centre of a circle and terminated both ways by the circumference is called a diameter of the circle.

INFERENCES FROM THE DEFINITION OF A CIRCLE.

The following simple properties of the circle are immediate inferences from these definitions :—

(1) *A circle is a closed figure.*

(2) *Any straight line drawn through a point within a circle and produced far enough in both directions will cut the circumference in two points.*

(3) *All diameters of a circle are equal.*

(4) *Each diameter of a circle is bisected at the centre.*

(5) *A point lies within, on, or without a circle, according as its distance from the centre is less than, equal to, or greater than the radius.*

(6) *The distance of a point from the centre of a circle is less than, equal to, or greater than the radius, according as it lies within, on, or without the circle.*

(7) *Circles of equal radii are identically equal.*

(8) *Circles that are identically equal have equal radii.*

CHORDS AND ARCS OF CIRCLES.

Def. 48. A chord of a circle is a straight line joining any two points on the circumference.

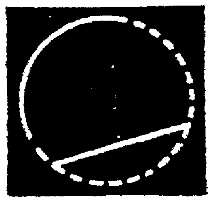


Fig. 223.

Def. 49. An arc of a circle is any part of the circumference.

Of the two arcs into which the circumference of a circle is divided by a chord which is not a diameter, the greater is called the major arc and the less the minor arc, and the two are said to be conjugate to one another.

Def. 50. A sector of a circle is a figure bounded by two radii and the arc intercepted between them.

Fig. 224.

When the bounding radii of a sector are at right angles to one another, the sector is a quarter of the circle and is called a quadrant.

ON SYMMETRY.

Def. 51. A figure is said to be symmetrical about a line if the line divides it into two parts which coincide when the figure is folded

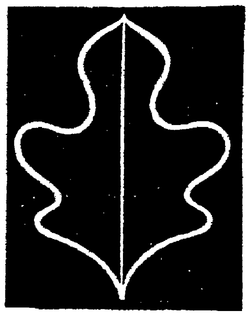


Fig. 225.

about the line. The line is called the **axis of symmetry** of the figure.

If a figure is symmetrical about a line, one-half of the figure is called the **image** or **reflection** of the other half in that line.

A point and its image are said to be **symmetrically opposite** with regard to the axis of symmetry.

Def. 52. A figure is said to be symmetrical about a point if the point bisects every straight line drawn through it to meet the

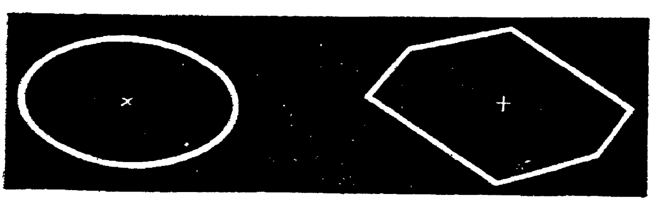


Fig. 226. Fig. 227.

boundary of the figure in both directions. The point is called the centre of symmetry of the figure.

THEOREM 34.

Gen. Enun. *A circle is symmetrical about any diameter.*

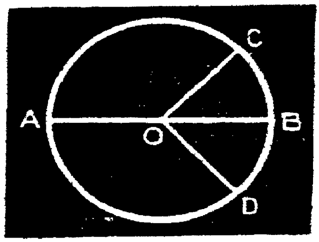


FIG. 228.

Part. Enun. Let AB be any diam. of a ⊙ whose centre is O.
It is reqd. to prove that
the ⊙ *is symmetrical about* AB.

Const. Suppose radii OC, OD drawn such that $\angle$ BOC = $\angle$ BOD.

Proof. Fold the fig. about AB until the arc ACB lies on the same side of AB as the arc ADB.

Then ∵ $\angle$ BOC = $\angle$ BOD,
∴ OC will lie along OD.
And ∵ OC = OD,
∴ C will coincide with D.

Similarly it can be shown that any other pt. on the arc ACB will coincide with some pt. on the arc ADB, and that every pt. on the arc ADB will coincide with some pt. on the arc ACB.

∴ arc ACB coincides with arc ADB.
∴ the ⊙ is symmetrical about AB. Q.E.D.

Cor. 1. *A diameter of a circle bisects at right angles the chord joining points symmetrically opposite with regard to that diameter.*

Const. Draw CD (Fig. 228) cutting AB at E.

Proof. ∵ C coincides with D on folding about AB.
∴ EC coincides with ED and is equal to it.
Also $\angle$ OEC coincides with $\angle$ OED and is equal to it.
But these are adjacent $\angle$s.
∴ AB bisects CD at rt. $\angle$s. Q.E.D.

Cor. 2. *The line through the centres of two circles* (*called their* **line of centres**) *is an axis of symmetry.*

For the line of centres contains a diam. of each ⊙.

Exercises.

711. A st. line CD bisects a st. line AB at rt. $\angle$s; prove that the pt. A is the image of the pt. B in the line CD.

712. *A pt.* A *is the image of a pt.* B *in a line* CD; *prove that* A *and* B *are equidistant from any pt. in* CD.

713. A pt. A is the image of a pt. B in a line CD; prove that CD bisects the $\angle$ formed by joining A and B to any pt. in CD.

714. A st. line AB is the image of a st. line CD in a line EF; prove that AB and CD are either ‖ to EF or, when produced, meet in EF.

* 715. *A st. line* AB *is the image of a st. line* CD *in a line* EF; *prove that* AD *and* BC *intersect on* EF.

* 716. AC, AD are chords of a $\odot$ equally inclined to the diam. AB; prove that AC = AD.

* 717. *A* $\odot$ *is divided into* 4 = *parts by* 2 *diams. at rt.* $\angle$*s to one another.*

* 718. *If* 2 $\odot$*s intersect, their line of centres bisects their common chord at rt.* $\angle$*s.*

* 719. Two = $\odot$s intersect so that the centre of each is on the other; prove that the sq. on the radius of either $\odot$ = ⅓ the sq. on the common chord.

* 720. *An isos.* $\triangle$ *is symmetrical about the bisector of the vert.* $\angle$.

* 721. Prove by symmetry that the bisectors of the base $\angle$s of an isos. $\triangle$ meet on the bisector of the vert. $\angle$.

* 722. Prove by symmetry that the medians of an isos. $\triangle$ are concurrent.

723. *A* $\odot$ *is symmetrical about its centre.*

724. A sq. is symmetrical about a pt.

725. Describe and prove the symmetry of a rect.

726. Which of the following letters have axial symmetry and which have central symmetry?—

B, E, H, M, O, S, W, Z.

* 727. Is a ▱gm symmetrical about a pt.?

* 728. Three = $\odot$s pass through a pt. O and intersect, 2 and 2, in 3 pts. A, B and C; prove that OA, OB, OC are $\perp$s to the sides of the $\triangle$ ABC. (Mad. Mat.)

THEOREM 35.

(Euc. III. 3.)

Gen. Enun. (A) *A straight line drawn from the centre of a circle to bisect a chord which is not a diameter is at right angles to the chord.*

Conversely,

(B) *The perpendicular to a chord of a circle from the centre bisects the chord.*

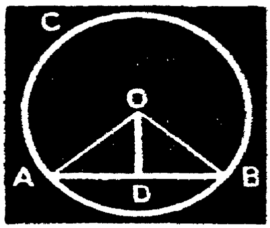


FIG. 229.

Part. Enun. Let **OD** be a st. line drawn from **O** the centre of the $\odot$ **ABC** to **D** a pt. in the chord **AB** which is not a diam.

It is reqd. to prove

(A) *If* **AD = BD**, *then* **OD** *is* $\perp$ *to* **AB**.

Conversely,

(B) *If* **OD** *is* $\perp$ *to* **AB**, *then* **AD = BD**.

Const. Join **OA, OB**.

Proof of (A). In the $\triangle$s **AOD, BOD**

$$\because \begin{cases} AD = BD & \text{. . . . . . Hyp.} \\ OD \text{ is common.} \\ \text{radius } OA = \text{radius } OB. \end{cases}$$

$\therefore$ $\triangle AOD \equiv \triangle BOD$ . . . Th. 14.

$\therefore$ $\angle ADO = \angle BDO$.

$\therefore$ **OD** is $\perp$ to **AB**. Q.E.D.

Proof of (B). In the $\triangle$s **AOD, BOD**

$$\because \begin{cases} \text{rt. } \angle ADO = \text{rt. } \angle BDO & \text{. . . Hyp.} \\ OD \text{ is common.} \\ \text{radius } OA = \text{radius } OB. \end{cases}$$

$\therefore$ $\triangle AOD \equiv \triangle BOD$ . . . Th. 15.

$\therefore$ **AD = BD**. Q.E.D.

Cor. *The perpendicular bisector of any chord of a circle passes through the centre.*

Exercises.

729. *The $\perp$ bisectors of any 2 chords of a $\odot$ intersect in the centre.*

730. P, Q, R, S, T are pts. on a $\odot$; prove that the $\perp$ bisectors of PQ, QR, RS, ST, TP are concurrent.

731. *A $\odot$ cannot have 2 centres.* (Punj. Mat.)

732. *The locus of the centres of $\odot$s passing through 2 fixed pts. is the $\perp$ bisector of the st. line joining the 2 pts.*

733. If 2 $\odot$s are concentric they cannot have a common pt. (Allah Mat.)

734. The $\perp$ bisector of either of 2 $\parallel$ chords of a $\odot$ is also the $\perp$ bisector of the other.

735. *The mid. pts. of several $\parallel$ chords of a $\odot$ lie on that diam. of the $\odot$ which is $\perp$ to them all.*

736. If the vertices of a trapezium lie upon a $\odot$ it must be isos.

737. *Two chords of a $\odot$ which are not both diams. cannot bisect each other.* (Euc. III. 4.)

738. PQRS is a $\square$gm whose vertices lie upon a $\odot$; prove that PR, QS are diams. of the $\odot$.

739. *Every $\odot$ passing through a fixed pt. and having its centre on a fixed st. line must pass through another fixed pt.*

740. If 2 $\odot$s intersect in one pt. they must intersect in another.

741. *If 2 $\odot$s have a common pt. they cannot have the same centre.* (Euc. III. 5, 6.)

742. If a st. line is drawn to cut 2 concent. $\odot$s its intercepts between the $\odot$s are =.

* 743. *The st. line joining any 2 pts. on a $\odot$ falls within the $\odot$.* (Euc. III. 2.)

* 744. *Through a pt. of intersection of 2 $\odot$s whose centres are* P, Q *a st. line* AB *is drawn $\parallel$ to* PQ *and terminated by the $\odot$s; prove that* AB = 2PQ.

* 745. Through the pts. of intersection of 2 $\odot$s st. lines AB, CD are drawn $\parallel$ to one another and meeting the $\odot$s again in A, B, C, D; prove that AB = CD.

* 746. Through a pt. of intersection of 2 $\odot$s st. lines PQ, RS are drawn equally inclined to the common chord and meeting the $\odot$s again in P, Q, R, S; prove that PQ = RS.

* 747. The greatest st. line that can be drawn through a pt. of intersection of 2 $\odot$s terminated by the $\odot$s is $\parallel$ to the line of centres. (Mad. F.E.)

* 748. Two st. lines OPQ, ORS are drawn from an external pt. O to cut a given $\odot$ PQSR; prove that the intersection of PS and QR cannot be the centre of the $\odot$.

Def. 53. A circle is said to be circumscribed about a rectilineal figure, and the figure is said to be inscribed in the circle if the circle

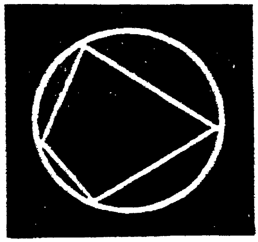


FIG. 230.

passes through all the vertices of the rectilineal figure.

The circumscribed circle of a triangle is called the circumcircle, its centre is called the circumcentre and its radius the circum-radius of the triangle.

Def. 54. Points are said to be concyclic when they lie on the same circle.

THEOREM 36.

(Euc. III. 10.)

Gen. Enun. *There is one circle, and one only, which passes through three given points not in a straight line.*

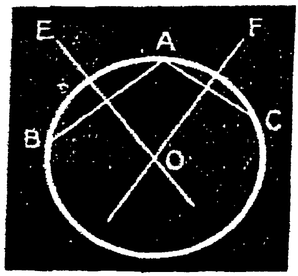


FIG. 231.

Part. Enun. Let A, B, C be 3 pts. not in a st. line.
It is reqd. to prove that
one ⊙, *and one only, will pass through* A, B *and* C.

Const. Join AB, AC.
Let EO, FO, the perp. bisectors of AB, AC respy., meet in O (see Ex. 44).

Proof. ∵ EO is the locus of pts. equidistant from A and B. Th. 24.
And FO is the locus of pts. equidistant from A and C. Th. 24.
∴ O, the pt. common to EO, FO, is equidistant from A, B and C.
And O is the only pt. common to EO, FO.
∴ a ⊙ with centre O and radius OA, and this ⊙ only, will pass through A, B and C. Q.E.D.

Exercises.

749. *If 2 ⊙s have more than 2 pts. common they must coincide.* (Euc. III. 10.)

750. Two ⊙s cannot have a common arc.

751. A ⊙ cannot be made to pass through 3 given pts. in the same st. line.

752. *A st. line cannot cut a ⊙ in 3 or more pts.*

753. *The ⊥ bisectors of the sides of a △ meet in a pt.*

* 754. *If a pt. is equidistant from more than 2 pts. on a ⊙ it is the centre of that ⊙.* (Euc. III. 9.)

* 755. PQRS is a quadl. whose diags. intersect at T; prove that the circum-centres of the △s PQT, QRT, RST, SPT are the angular pts. of a ▱gm.

* 756. A ⊙ can be circumscribed about any isos. trapezium.

THEOREM 37.

(Euc. III. 26, 27.)

Gen. Enun. *In equal circles (or in the same circle):—*
(A) *If two arcs subtend equal angles at the centres, they are equal.*
Conversely,
(B) *If two arcs are equal they subtend equal angles at the centres.*

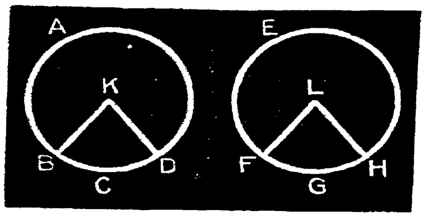


FIG. 232.

Part. Enun. Let BCD, FGH be arcs of = ⊙s ABD, EFH whose centres are K, L respy.
It is reqd. to prove
(A) *If* ∠ BKD = ∠ FLH, *then arc* BCD = *arc* FGH.
Conversely,
(B) *If arc* BCD = *arc* FGH, *then* ∠ BKD = ∠ FLH.

Proof of (A). Apply ⊙ ABD to ⊙ EFH so that K falls on L and KB lies along LF.
Then ∵ KB = LF . . . . . Hyp.
∴ B falls on F.
And ∵ ∠ BKD = ∠ FLH . . . Hyp.
∴ KD lies along LH.
And ∵ KD = LH . . . . . Hyp.
∴ D falls on H.
Now the whole circumference ABCD coincides with the whole circumference EFGH (∵ the ⊙s are =).
∴ arc BCD coincides with arc FGH.
∴ arc BCD = arc FGH. Q.E.D.

Proof of (B). Apply ⊙ ABD to ⊙ EFH so that K falls on L and KB lies along LF.
Then ∵ KB = LF . . . . . Hyp.
∴ B falls on F.
Now the whole circumference ABCD coincides with the whole circumference EFGH (∵ the ⊙s are =).

And arc BCD = arc FGH . . . Hyp.
$\therefore$ arc BCD coincides with arc FGH.
$\therefore$ D falls on H.
$\therefore$ $\angle$ BKD coincides with $\angle$ FLH.
$\therefore$ $\angle$ BKD = $\angle$ FLH. Q.E.D.

Exercises.

757. *In = $\odot$s (or in the same $\odot$) of 2 unequal $\angle$s at the centres the greater $\angle$ stands upon the greater arc.*

758. *In = $\odot$s (or in the same $\odot$) 2 sectors are = if the $\angle$s between their bounding radii are =.*

759. PQ, PR are = chords in a $\odot$; prove that P is the mid. pt. of the arc QPR.

760. C is the mid. pt. of an arc ACB; prove that C is equidistant from the radii through A and B.

* 761. AB, CD are diams. of a $\odot$ and CE is a chord $\parallel$ to AB; prove that B is the mid. pt. of the arc DBE.

* 762. DF is the diam. and EG a chord of a semi-$\odot$ DEGF; DF and EG are produced to meet at H; if GH = $\frac{1}{2}$DF, prove that arc FG = $\frac{1}{3}$ arc DE.

THEOREM 38.

(Euc. III. 28, 29.)

Gen. Enun. *In equal circles (or in the same circle):—*
(A) *If two chords are equal they cut off equal arcs.*
Conversely,
(B) *If two arcs are equal the chords of the arcs are equal.*

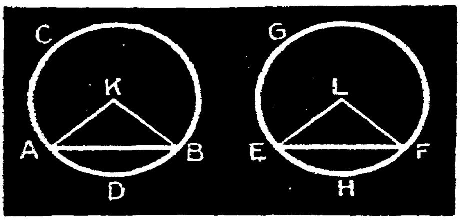


FIG. 233.

Part. Enun. Let AB, EF be chords of = ⊙s ABC, EFG whose centres are K, L respy.

It is reqd to prove

(A) *If chord* AB = *chord* EF, *then minor arc* ADB = *minor arc* EHF *and major arc* ACB = *major arc* EGF.

Conversely,

(B) *If arc* ADB = *arc* EHF, *then chord* AB = *chord* EF.

Const. Join KA, KB, LE, LF.

Proof of (A). In the △s KAB, LEF

∵ { KA = LE	Hyp.
KB = LF	Hyp.
AB = EF . . . . . . }	Hyp.
∴ △ KAB ≡ △ LEF	Th. 14.
∴ ∟ AKB = ∟ ELF.	
∴ minor arc ADB = minor arc EHF .	Th. 37.
But whole circumference ADBC = whole circumference EHFG	Hyp.
∴ major arc ACB = major arc EGF.	Q.E.D.

Proof of (B). ∵ arc ADB = arc EHF	Hyp.
∴ ∟ AKB = ∟ ELF	Th. 37.

Hence in the △s KAB, LEF,

$$\because \begin{cases} KA = LE \quad . \quad . \quad . \quad . \quad . & \text{Hyp.} \\ KB = LF \quad . \quad . \quad . \quad . \quad . & \text{Hyp.} \\ \angle AKB = \angle ELF & \end{cases}$$

$\therefore \triangle KAB \equiv \triangle LEF$ . . . . . Th. 10.

$\therefore$ chord AB = chord EF. Q.E.D.

Exercises.

763. *In = ⊙s (or in the same ⊙) = chords subtend = ∠s at the centres.*

764. If in 2 ⊙s = chords subtend = ∠s at the centres, the ⊙s must be =.

765. If 2 opp. sides of a quadl. inscd. in a ⊙ are =, its diags. are =.

766. *The st. lines joining alternate angular pts. of a regular polygon inscd. in a ⊙ are =.*

* 767. *An equilat. polygon inscd. in a ⊙ is also equiang.*

* 768. ABC is a △ inscd. in a ⊙ and D, E are the mid. pts. of the minor arcs cut off by AB, AC; prove that DE is equally inclined to AB and AC.

THEOREM 39.

(Euc. III. 14, 15.)

Gen. Enun. (A) *Equal chords of a circle are equidistant from the centre, and the greater chord of a circle is nearer the centre than the less.*

Conversely,

(B) *Chords of a circle that are equidistant from the centre are equal, and a chord of a circle nearer to the centre is greater than one more remote.*

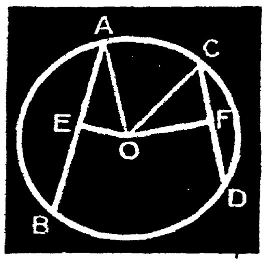


FIG. 234.

Part. Enun. Let AB, CD be chords of a ⊙ ABC whose centre is O, and let OE, OF be the distances of O from AB, CD respy.

It is reqd. to prove

$$\text{(A) } \textit{If } AB \overset{=}{\gtrless} CD, \textit{ then } OE \overset{=}{\lessgtr} OF.$$

Conversely,

$$\text{(B) } \textit{If } OE \overset{=}{\gtrless} OF, \textit{ then } AB \overset{=}{\lessgtr} CD.$$

Const. Join OA, OC.

Proof.	$\because \llcorner$s OEA, OFC are rt. $\llcorner$s	Def. 38.
	$\therefore OA^2 = OE^2 + EA^2$	Th. 32.
	And $OC^2 = OF^2 + FC^2$	Th. 32.
	But $OA^2 = OC^2$ ($\because OA = OC$).	
	$\therefore OE^2 + EA^2 = OF^2 + FC^2$.	

Hence if $EA^2 \overset{=}{\gtrless} FC^2$, then $OE^2 \overset{=}{\lessgtr} OF^2$.

And conversely,

$$\text{if } OE^2 \overset{=}{\underset{<}{>}} OF^2, \text{ then } EA^2 \overset{=}{\underset{>}{<}} FC^2.$$

$$\therefore \text{ if } EA \overset{=}{\underset{<}{>}} FC, \text{ then } OE \overset{=}{\underset{>}{<}} OF.$$

And conversely,

$$\text{if } OE \overset{=}{\underset{<}{>}} OF, \text{ then } EA \overset{=}{\underset{>}{<}} FC.$$

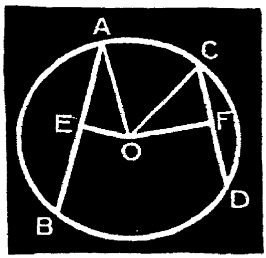


FIG. 284.

But $EA = \frac{1}{2}AB$ and $FC = \frac{1}{2}CD$ . . . Th. 35.

$$\therefore \text{ if } AB \overset{=}{\underset{<}{>}} CD, \text{ then } OE \overset{=}{\underset{>}{<}} OF.$$

And conversely,

$$\text{if } OE \overset{=}{\underset{<}{>}} OF, \text{ then } AB \overset{=}{\underset{>}{<}} CD. \qquad \text{Q.E.D.}$$

Exercises.

769. *In = $\odot$s = chords are equidistant from the centres.*

770. *In = $\odot$s chords equidistant from the centres are =.*

771. In = $\odot$s (or in the same $\odot$) if 2 chords subtend = $\angle$s at the centres they are equidistant from the centres.

772. In = $\odot$s (or in the same $\odot$) if 2 chords are equidistant from the centres they subtend = $\angle$s at the centres.

773. *Of all chords that can be drawn through a given pt. within a $\odot$ the longest is the diam. through that pt.*

774. The pts. of intersection of any diam. with = and $\parallel$ chords in a $\odot$ are equidistant from the centre.

775. AB, CD are = chords in the greater of 2 concent. ⊙s; prove that the parts of AB, CD intercepted by the smaller ⊙ are =.

776. PQ, RS are = chords of a ⊙ intersecting at T; prove that they are equally inclined to the diam. through T.

777. *A chord of constant length slides round a fixed ⊙; prove that the locus of its mid. pt. is a concent. ⊙.*

778. *A chord of constant length slides round a fixed ⊙; prove that the locus of any pt. fixed in the chord is a concent. ⊙.*

779. *Of all chords that can be drawn through a given pt. within a ⊙ the shortest is the ⊥ to the diam. through that pt.*

780. *Equal chords of a ⊙ that intersect divide each other into = segments, the greater = the greater and the less = the less.*

781. Chords of a ⊙ that intersect so that a segment of the one = a segment of the other are =.

782. P, Q are the centres of 2 = ⊙s and ABCD is a st. line ∥ to PQ and meeting the ⊙s in A, B, C, D; prove that AB = CD.

783. P, Q are the centres of 2 = ⊙s and ABCD is a st. line through the mid. pt. of PQ and meeting the ⊙s at A, B, C, D; prove that AB = CD.

784. P, Q are the mid. pts. of 2 = chords in a ⊙ and the st. line joining P, Q is produced both ways to meet the ⊙ in R, S; prove that PR = QS.

* 785. PQ, RS are a chord and a diam. of a ⊙ not intersecting one another; prove that the sum of the ⊥s from R, S upon PQ is const. for all positions of RS.

* 786. PQ, RS are a chord and a diam. of a ⊙ intersecting one another; prove that the difference of the ⊥s from R, S upon PQ is const. for all positions of RS.

* 787. If 3 = chords of a ⊙ meet in the same pt., that pt. must be the centre of the ⊙.

* 788. AB, AC are = st. lines drawn from an external pt. A to the concave circumference of a ⊙; if AB, AC cut the convex circumference at D, E, prove that AD = AE.

* 789. Of all chords of a ⊙ which are bisected by a fixed chord the greater is that which meets the fixed chord at a pt. nearer its mid. pt.

TANGENCY.

Def. 55. A secant of a circle is a straight line of indefinite length which meets the circumference of the circle in two points.

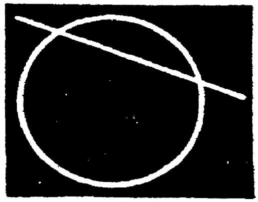


FIG. 235.

Def. 56. A tangent to a circle is a straight line which meets the circumference of the circle in one, and only one, point, however far

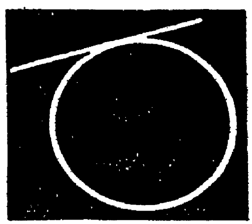


FIG. 236.

produced. This straight line is said to touch the circle, and the point where it meets the circle is called the point of contact.

Since the path of a moving point is a line, a secant of a circle may

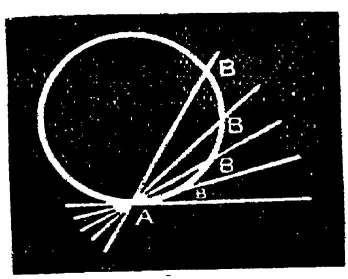


FIG. 237.

be regarded as a straight line passing through two different positions of a point moving in a circle. If the two positions are *consecutive*, that is to say, if there is no distance between them, the secant becomes a tangent, because it passes through two points which coincide, and does not meet the circle in another point however far produced. For example, suppose a secant, cutting a circle at A and B, to turn round A so that B continually approaches A. These points will ultimately coincide, and then the secant becomes a tangent to the circle. See Fig. 237.

Or, again, suppose a secant cutting a circle at A and B to move parallel to its original position. A and B will continually approach

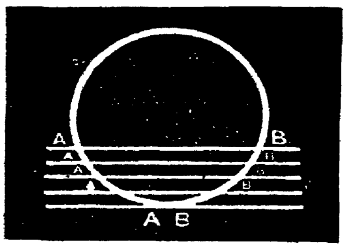


FIG. 238.

one another and ultimately coincide, so that the secant becomes a tangent to the circle. See Fig. 238.

Hence a tangent to a circle may be regarded as the "limiting" position of a secant when the points of intersection move into coincidence, and this suggests the following alternative definition of a tangent :—

Def. 57. A tangent to a circle is a straight line which meets the circumference of the circle in two consecutive points.

The advantage of this definition over Def. 56 is that it applies to all

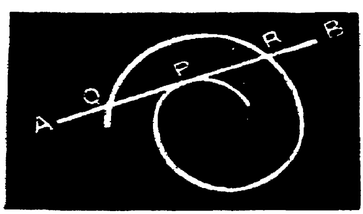


FIG. 239.

curves, as may be seen from Fig. 239, where AB is a tangent to a curve at P, though it also meets the curve at Q and R.

Def. 58. Two circles which meet in one, and only one, point are said to touch one another, and the point at which they meet is called the point of contact.

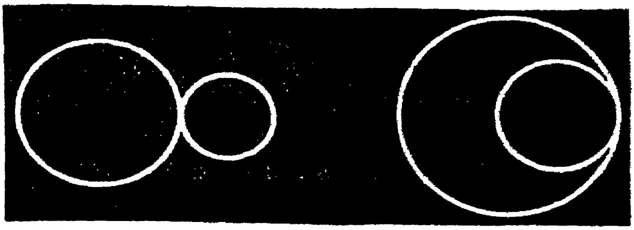


FIG. 240. FIG. 241.

We may regard one circle touching another as the limiting position of one circle cutting another, when the points of intersection move into coincidence; hence, as an alternative definition, we have :—

Def. 59. Two circles which meet in two consecutive points are said to touch one another.

This alternative definition applies to contact between any two curves, as shown in Fig. 242, where the curve **AB** touches the curve **CD** at **P**, though it also meets it at **Q**.

FIG. 242.

We infer from this alternative definition :—

If two circles touch one another they have the same tangent at their point of contact.

For suppose two circles to intersect and **AB** to be their common chord, as in Fig. 243. If the smaller circle is made to turn about **A**, **B** will move into coincidence with **A**, when the circles will touch one another and **AB** will become a common tangent at their point of contact. By turning in one direction about **A**, the smaller circle is made to touch the larger circle *externally*, as in Fig. 244; by turning

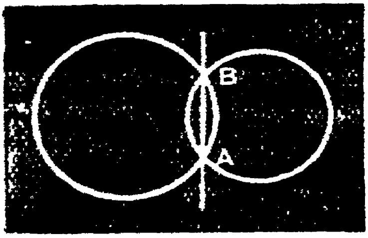


Fig. 243.

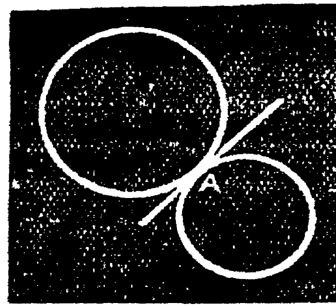


Fig. 244.

Fig. 245.

in the opposite direction it is made to touch the larger circle *internally*, as in Fig. 245.

Def. 60. Common tangents to two circles are said to be direct (Fig. 246) or transverse (Fig. 247) according as the two circles lie on the same side or on opposite sides of the common tangent.

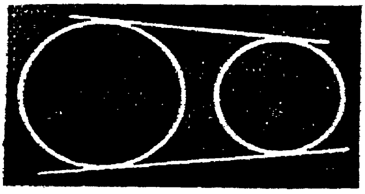


Fig. 246.

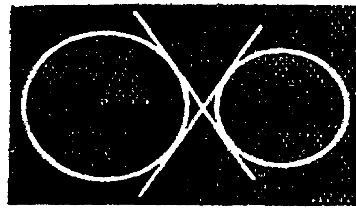


Fig. 247.

Def. 61. Two circles are said to cut orthogonally when the two tangents at either point of intersection are at right angles to one another.

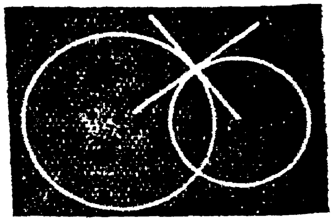


FIG. 248.

Def. 62. A circle is said to be inscribed in a rectilineal figure and the figure is said to be circumscribed about the circle if the circle touches all the sides of the rectilineal figure.

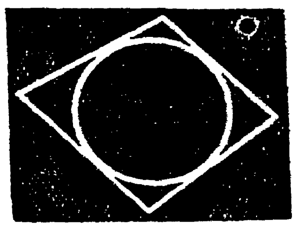


FIG. 249.

The inscribed circle of a triangle is called the **in-circle**, its centre is called the **in-centre**, and its radius the **in-radius** of the triangle.

If a circle touches one side of a triangle and the other two sides

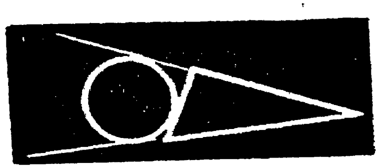


FIG. 250.

produced it is said to be **escribed** to the triangle and is called an **e-circle**, its centre is called an **e-centre**, and its radius an **e-radius** of the triangle.

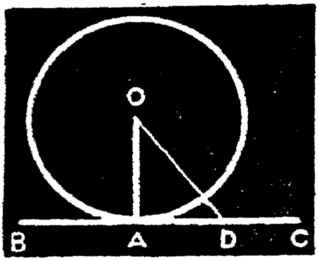


FIG. 251.

THEOREM 40.

(Euc. III. 18.)

Gen. Enun. *The tangent at any point of a circle and the radius through the point are perpendicular to one another.*

Part. Enun. Let **BC** be a tangent at a pt. **A** to a ⊙ whose centre is **C**

It is reqd. to prove that

radius **OA** *is* ⊥ *to tangent* **BC**.

Const. Take any pt. **D** in **BC** other than **A**. Join **OD**.

Proof. ∵ pt. **D** lies outside the ⊙.
∴ radius **OA** < **OD**.
Similarly **OA** < any other st. line from **O** to **BC**.
∴ **OA** is the shortest st. line from **O** to **BC**.
∴ radius **OA** is ⊥ to tangent **BC** . . . Th. 21.

Q.E.D.

Cor. 1. *Only one tangent can be drawn to a circle at any point on its circumference.*

Cor. 2. *The perpendicular to a diameter of a circle at either of its extremities is a tangent to the circle* (Euc. III. 16).

Cor. 3. *The perpendicular to a tangent to a circle at the point of contact passes through the centre* (Euc. III. 19).

Cor. 4. *The perpendicular to a tangent to a circle from the centre passes through the point of contact.*

Cor. 5. *If two tangents are drawn to a circle from an external point* (**A**) *the tangents are equal,* (**B**) *they subtend equal angles at the centre of the circle,* (**C**) *they make equal angles with the straight line joining the given point to the centre.*

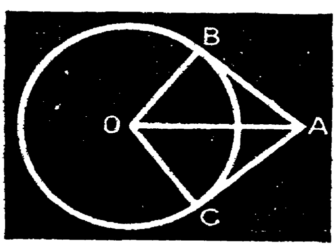


FIG. 251A.

Proof. In the $\triangle^s$ OBA, OCA.

$\because \begin{cases} OB = OC \text{ (radii)} \\ OA \text{ is common.} \\ \text{rt.} \angle OBA = \text{rt.} \angle OCA \end{cases}$ . . . Th. 40.

$\therefore \triangle OBA \equiv \triangle OCA$ . . . . Th. 15.

So that $\begin{cases} (A)\ AB = AC. \\ (B)\ \angle AOB = \angle AOC. \\ (C)\ \angle OAB = \angle OAC. \end{cases}$ Q.E.D.

ON THE METHOD OF LIMITS.

Since a tangent is only a particular kind of secant and touching circles a special case of cutting circles, we can deduce by what is known as the method of limits many of the properties of tangents and of touching circles from the properties of secants and of cutting circles respectively.

Proof of Theorem 40 by the Method of Limits :—

Let AB be a secant cutting a ⊙ whose centre is O at A and B, and let C be the mid. pt. of AB.

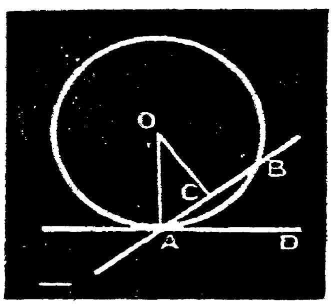


FIG. 252.

Join OA, OC.

Now suppose the secant AB to turn round A so that B moves into coincidence with A, and let AD be the ultimate position of AB.

Then AD is the tangent at A.

But when B coincides with A, C must also coincide with A, because C is the mid. pt. of AB.

Hence when B coincides with A, and the secant AB becomes the tangent AD, OC will coincide with OA.

But OC is $\perp$ to AB . . . . Th. 35.

$\therefore$ OA is $\perp$ to AD. Q.E.D.

Note.—Corollaries 1, 2, 3, 4 of Theorem 40 can all be proved by this method and afford valuable exercises.

Exercises.

790. *The tangents to a ⊙ at the extremities of a diam. are ∥.*

791. State and prove the converse of Ex. 790.

792. A st. line cannot touch a ⊙ in 2 or more pts.

793. The ⊥ to a diam. of a ⊙ at its extremity falls without the ⊙. (Euc. III. 16.)

794. PQ, PR are tangents to a ⊙ whose centre is O; prove that ∠ RPQ + ∠ ROQ = 180°.

795. *AB and AC are two tangents to a ⊙ whose centre is O; prove that AO bisects BC at rt. ∠s.*

796. *Any diam. of a ⊙ is the ⊥ bisector of all chords ∥ to the tangents at its extremities.*

797. *All chords of the greater of 2 concent. ⊙s which touch the lesser are =*

798. *Find the locus of the centres of ⊙s touching a given st. line at a given pt.*

799. *Two, and only two, tangents can be drawn to a ⊙ from an external pt.*

800. Any two tangents drawn to a ⊙ make = ∠s with the st. line joining their pts. of contact.

801. If P, Q and R, S are the pts. of contact of the 2 direct common tangents to 2 ⊙s, prove that PQ = RS.

802. If P, Q and R, S are the pts. of contact of the 2 transverse tangents to 2 ⊙s, prove that PQ = RS.

803. If 2 ⊙s intersect at P and Q, prove that the tangents to both ⊙s at P make with one another the same ∠ as the tangents to both ⊙s at Q.

804. *If a ⊙ can be inscd. in a quadl. the sum of one pair of opp. sides of the quadl. = the sum of the other pair.* (Bom. Sch. Fin.)

805. If a ⊙ can be inscd. in a ▱gm, the ▱gm is a rhombus.

806. If a ⊙ can be inscd. in a hexagon, the sum of alternate sides of the hexagon = $\frac{1}{2}$ its perim.

807. *If 2 ⊙s cut each other orthogonally, the radii to each pt. of intersection are at rt. ∠s to one another.*

808. If 2 ⊙s cut each other orthogonally, the sum of the squares on their radii = the sq. on the st. line joining their centres. (Punj. Mat.)

809. If P, Q are the pts. of contact of a common tangent to 2 ⊙s, these ⊙s are cut orthogonally by the ⊙ descd. on PQ as diam.

810. *If a ⊙ can be inscd. in a polygon, the area of the polygon = rect contained by ½ its perim. and the radius of the ⊙.*

811. *If a st. line touches a ⊙, any other st. line drawn through the pt. of contact will cut the ⊙.*

812. *All tangents to the lesser of 2 concent. ⊙s. from pts. on the greater are =.*

813. *Find the locus of the centre of a ⊙ of given radius touching a given st. line.*

814. *Find the locus of the centre of a ⊙ touching 2 given ∥ st. lines.*

815. *Find the locus of the centre of a ⊙ touching 2 given intersecting st. lines.*

816. The line of centres of 2 ⊙s passes through the intersection of their direct common tangents.

817. The line of centres of 2 ⊙s passes through the intersection of their transverse common tangents.

818. *One ⊙, and only one, can be inscd. in a given △.*

819. *Three ⊙s, and only three, can be descd. to touch one side of a △ and the other 2 sides produced.*

820. The intersection of the diags. of a rhombus is the centre of the inscd. ⊙.

821. If a ⊙ can be inscd. in a quadl., the ∠s at the centre subtended by each pair of opp. sides are supplementary.

822. If a ⊙ can be inscd. in a hexagon, the ∠s at the centre subtended by any 3 alternate sides are together = 2 rt. ∠s.

* 823. PQ, PR are tangents to a ⊙ and QS is a diam.; prove that ∠ QPR = 2 ∠ RQS.

* 824. AB, CD are diams. of a ⊙ whose centre is O, and tangents are drawn to the ⊙ at A, B, C, D; prove that the diags. of the quadl. so formed intersect at O.

* 825. In the greater of 2 concent. ⊙s 2 chords PQ and PR are drawn to touch the lesser at S and T; prove that ST = ½QR.

* 826. PQ is a chord and PR a diam. of a ⊙; PS bisects ∠ RPQ and meets the ⊙ at S; prove that PQ is ⊥ to the tangent to the ⊙ at S.

* 827. *The in-⊙ of a △ ABC touches BC at D; prove that the difference between BD and CD = the difference between AB and AC.*

* 828. *Two tangents are drawn to a ⊙ from a fixed pt. and produced to meet a third tangent; prove that the part of the third tangent that they intercept subtends a const. ∠ at the centre of the ⊙.*

*829. A tangent to a ⊙ meets two ∥ tangents at P and Q; prove that PQ subtends a rt. ∟ at the centre. (Bom. Sch. Fin.)

*830. Prove that the in-centre, the circum-centre and the vertex of an isos. △ are collinear.

*831. *If* I *is the in-centre of a* △ *and* I_1, I_2, I_3 *its e-centres, prove that* I *is the orthocentre of the* △ $I_1 I_2 I_3$. (Bom. Prev.)

*832. In Fig. 253 prove that :—

(i) $AE_1 = AF_1 = BF_2 = BD_2 = CD_3 = CE_3 = s.$

(ii) $AE = AF = s - a$; $BF = BD = s - b$; $CD = CE = s - c$.

(iii) $BD_3 = BF_3 = CD_2 = CE_2 = s - a.$
$CD_1 = CE_1 = AE_3 = AF_3 = s - b.$
$AE_2 = AF_2 = BD_1 = BF_1 = s - c.$

(iv) $AE = CE_2$; $BF = AF_3$; $CD = BD_1$.

(v) $EE_1 = FF_1 = a$; $FF_2 = DD_2 = b$; $DD_3 = EE_3 = c$.

(vi) $D_1D_3 = c$; $E_2E_3 = a$; $F_3F_1 = b$.

(vii) $DD_1 = c - b$; $EE_2 = a - c$; $FF_3 = a - b$,

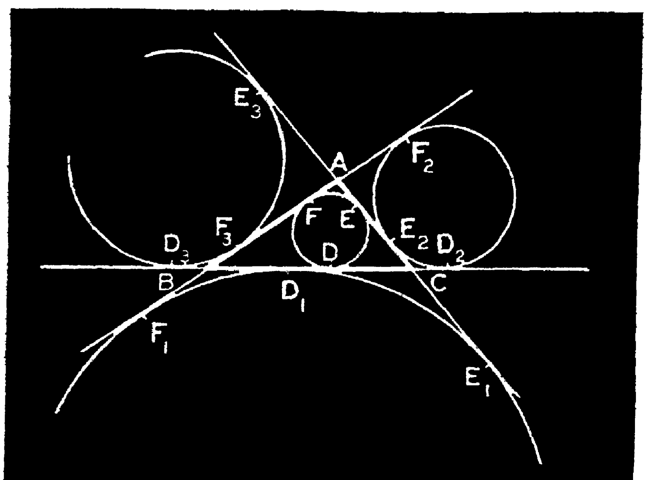


FIG. 253.

where $BC = a$, $CA = b$, $AB = c$, $s = \dfrac{a + b + c}{2}$.

THEOREM 41.

(Euc. III. 11, 12.)

Gen. Enun. *If two circles touch, the point of contact lies on their line of centres.*

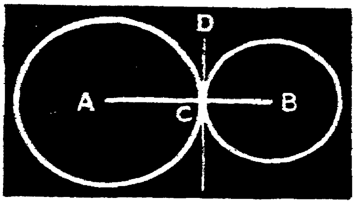


FIG. 254.

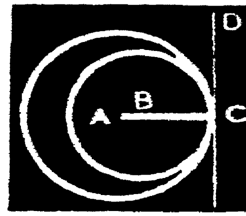


FIG. 255.

Part. Enun. Let 2 ⊙s whose centres are A and B touch at the pt. C.

It is reqd. to prove that

A, B, C *are in one st. line.*

Const. Suppose CD to be the common tangent to the ⊙s. Join AC, BC.

Proof. ∵ CD is a tangent to a ⊙ and AC is the radius of the ⊙ through the pt. of contact.

∴ ∟ ACD is a rt. ∟ . . . . . Th. 40.

Similarly ∟ BCD is a rt. ∟ . . . . Th. 40.

∴ A, B, C are in one st. line . . . . (Why?)

Q.E.D.

Note.—Theorem 41 can easily be deduced by the Method of Limits from the theorem:—

If two ⊙s intersect, their line of centres bisects their common chord (Ex. 718).

Exercises.

833. *If 2 ⊙s touch, the distance between their centres = the sum or difference of their radii.*

834. State and prove the converse of Ex. 833.

835. *Two ⊙s cannot touch one another at 2 pts.* (Euc. III. 13.)

836. If 2 ⊙s meet on their line of centres, they must touch each other.

837. A ⊙ of centre P touches externally 2 ⊙s of centres Q and R; prove that the difference between PQ and PR = the difference between the radii of the ⊙s of centres Q and R.

838. *Find the locus of the centre of a ⊙ of given radius touching a given ⊙ (1) externally, (2) internally.*

839. *Find the locus of the centre of a ⊙ touching a given ⊙ at a given pt.*

840. *Two = ⊙s touch externally; find the locus of the centre of a third ⊙ touching both.*

841. *If 2 ⊙s touch externally or internally and any st. line is drawn through the pt. of contact to cut the ⊙s again in 2 pts., the radii to these pts. are ∥.*

842. If 2 ⊙s touch externally or internally and any st. line is drawn through the pt. of contact to cut the ⊙s again in 2 pts., the tangents to the ⊙s at these pts. are ∥.

843. If 2 ⊙s touch externally or internally and 2 diams. are drawn ∥ to one another, the pt. of contact and an extremity of each diam. are collinear. (Bom. Sch. Fin.)

844. *Two = ⊙s touch externally; prove that they intercept = chords on any st. line drawn through the pt. of contact.*

* 845. Three ⊙s touch externally at P, Q, R; prove that the common tangents at P, Q, R meet in a pt. equidistant from P, Q, R.

* 846. Three ⊙s whose centres are A, B, C touch externally at P, Q, R; prove that the in-⊙ of △ ABC is the circum-⊙ of △ PQR.

* 847. A, B are the pts. of contact of a common tangent to 2 ⊙s which touch externally at C; prove that the line of centres of the ⊙s is a tangent to the ⊙ descd. on AB as diam. (Punj. Inter.)

* 848. A ⊙ with centre P touches a ⊙ with centre Q internally and a ⊙ with centre R touches one internally and the other externally; prove that PR + QR = a const. for all positions of R.

* 849. *Two ⊙s touch externally or internally at A and ∥ tangents are drawn, one to each ⊙ at B and C; prove that AB and AC are either in the same st. line or at rt. ∟s.*

* 850. Two = ⊙s with centres P and Q touch externally at R. Chords RS, RT are drawn, one in each ⊙, at rt. ∟s to one another. Prove that ST is = and ∥ to PQ.

* 851. Three ⊙s touch externally at P, Q, R. PQ, PR are produced to cut one of the ⊙s again in S, T. Prove that the line of centres of the other two ⊙s is ∥ to ST.

* 852. Two = ⊙s intersect at A, B and are touched externally by a third ⊙ whose centre is C; prove that A, B, C are collinear.

* 853. A ⊙ rolls within another ⊙ of twice its radius; prove that the locus of any pt. on the smaller ⊙ is a diam. of the larger.

ANGLE PROPERTIES OF CIRCLES.

Theorem 42.

(Euc. III. 20.)

Gen. Enun. *The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.*

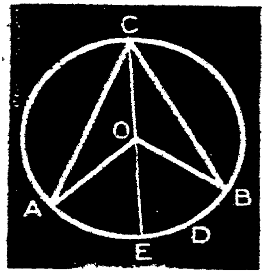


Fig. 256.

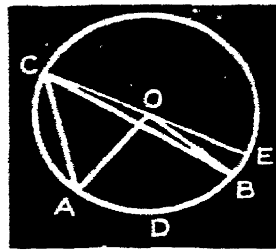


Fig. 257.

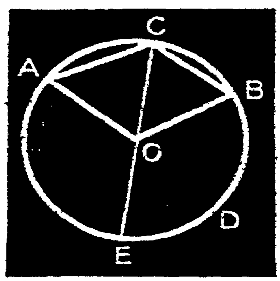


Fig. 258.

Part. Enun. Let ABC be a ⊙ of which O is the centre, and let ADB be an arc of the ⊙ ABC and C any pt. on the remaining part of the circumference.

It is reqd. to prove that

$$\angle AOB = 2\angle ACB.$$

Const. Join CO and produce it to E.

Proof. ∵ radius OA = radius OC.

∴ $\angle OAC = \angle OCA$ . . . . . Th. 12.

Also ext. $\angle EOA = \angle OAC + \angle OCA$. Th. 8, Cor 3.

$= 2\angle OCA.$

Similarly $\angle EOB = 2\angle OCB.$

∴ sum or difference of $\angle$s EOA, EOB = twice the sum or difference of $\angle$s OCA, OCB.

∴ $\angle AOB = 2\angle ACB.$ Q.E.D.

Exercises.

854. *In = $\odot$s (or in the same $\odot$) if 2 arcs subtend = $\angle$s at the circumferences they are =.*

855. *In = $\odot$s (or in the same $\odot$) if 2 arcs are = they subtend = $\angle$s at the circumferences.*

856. If C, D are pts. of trisection of an arc of a $\odot$ ACDB, prove that $\angle$ DAC = $\angle$ DAB.

857. *The internal bisector of the vert. $\angle$ of a $\triangle$ meets the circum-$\odot$ in a pt. equidistant from the ends of the base.*

858. *If a polygon inscd. in a $\odot$ is equilat. it is also equiang.* (Cal. Mat.)

859. *An equiang. polygon inscd. in a $\odot$ has its alternate sides =.*

860. *If an equiang. polygon inscd. in a $\odot$ has an odd number of sides it must be equilat.*

861. *Parallel chords in a $\odot$ intercept = arcs on the circumference.*

862. State and prove the converse of Ex. 861.

863. PQ, RS are = chords in a $\odot$ PQSR; prove that the quadl. PQSR is an isos. trapezium.

864. AB, CD are $\parallel$ chords in a $\odot$; prove that AC = BD and AD = BC.

865. PQ, PR are chords of a $\odot$ $\parallel$ respy. to 2 other chords *pq*, *pr* of the same $\odot$; prove that R*q* is $\parallel$ to Q*r*.

866. *Through* A, B, *the pts. of intersection of 2 $\odot$s, $\parallel$ st. lines* CAD, EBF *are drawn meeting the $\odot$s in* C, E *and* D, F; *prove that chord* CE = *chord* DF.

867. *Two chords intersect within a $\odot$; prove that the sum of the opp. arcs which they intercept = the arc intercepted by $\parallel$ chords meeting on the $\odot$.*

868. *Two chords intersect without a $\odot$; prove that the difference of the arcs which they intercept = the arc intercepted by $\parallel$ chords meeting on the $\odot$.*

869. Two chords intersect within a $\odot$; prove that they contain an $\angle = \frac{1}{2}$ the $\angle$ subtended at the centre by an arc = the sum of the opp. arcs which they intercept. (All. Mat.)

870. Two chords intersect without a $\odot$; prove that they contain an $\angle = \frac{1}{2}$ the $\angle$ subtended at the centre by an arc = the difference of the arcs which they intercept.

871. *Two = $\odot$s intersect in* A *and* B; *through* A *st. lines* CAD, EAF *are drawn meeting the $\odot$s in* C, D, E, F; *prove that chord* CE = *chord* DF.

* 872. E is the pt. of intersection of 2 chords AB, CD of a $\odot$ ACBD whose centre is O; prove that $\angle$ AOC + $\angle$ BOD = 2 $\angle$ AEC.

* 873. *If the in-$\odot$ of a $\triangle$* ABC *touches its sides at* D, E, F, *the $\angle$s of the $\triangle$* DEF *are* $90° - \frac{A}{2}$, $90° - \frac{B}{2}$, $90° - \frac{C}{2}$.

* 874. *Two = $\odot$s intersect in* A *and* B; *through* A *a st. line is drawn meeting the $\odot$s again in* C *and* D; *prove that* C *and* D *are equidistant from* B.

* 875. The internal bisector of the vert. ∠ A of a △ ABC meets the circum-⊙ in D ; DE, DF are drawn ⊥s to AB, AC respy. ; prove that AE = ½(AB + AC).

* 876. AB, AC are chords of a ⊙ and D, E are the mid. pts. of the arcs AB, AC ; the st. line joining D, E cuts AB, AC at F, G respy. ; prove that AF = AG.

* 877. ABCD is a square ; arcs of ⊙s AEC, BED are descd. with centres B, C respy. cutting one another at the pt. E within the sq. ; prove that ∠ BDE = ⅙ rt. ∠. (Mad. Mat.)

* 878. P, Q, R are pts. on a ⊙ and S, T, X are the mid. pts. of the arcs PQ, QR, RP respy. ; prove that the st. lines PT, QX, RS are concurrent.

* 879. Two = ⊙s cut one another at A, B so that the centre of each is on the other ; a st. line is drawn through A terminated by the ⊙s at C and D ; prove that CD = DB = BC.

* 880. If I and S are the in-centre and circum-centre respy. of a △ PQR, prove that ∠ SPI = ½ difference of ∠s Q and R.

* 881. ABCD and PQRS are quadls. inscd. in the same ⊙ having BC ∥ to QR and AD ∥ to PS ; prove that ∠ contained by AC and PR = ∠ contained by BD and QS.

Def. 63. A segment of a circle is the figure bounded by a chord of the circle and one of the arcs into which it divides the circumference.

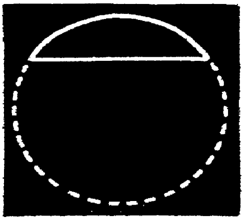


FIG. 259.

The chord of a segment is called its base.

A segment of a circle is called a semi-circle when its chord is a diameter of the circle.

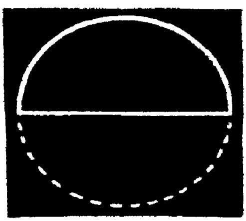


FIG 260.

A segment of a circle is called a major segment or a minor segment according as its arc is a major arc or a minor arc.

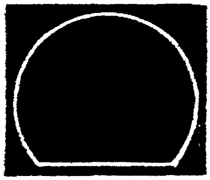


FIG. 261.

FIG. 262.

Def. 64. An angle in a segment is the angle subtended by the chord of the segment at a point on the arc.

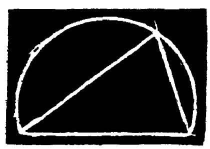


FIG. 263.

Theorem 43.

(Euc. III. 21.)

Gen. Enun. ***Angles in the same segment of a circle are equal.***

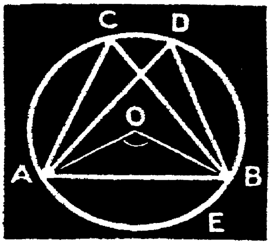


Fig. 264

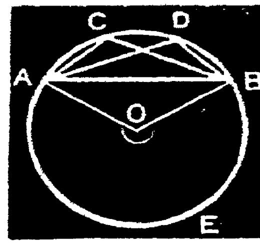


Fig. 265.

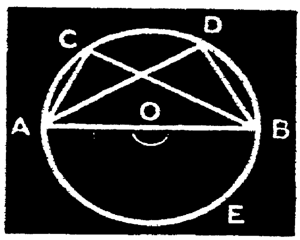


Fig. 266.

Part. Enun. Let ACB, ADB be ∟s in the same segment ACDB of a ⊙ ABC whose centre is O.

It is reqd. to prove that

$$\angle ACB = \angle ADB.$$

Const. Join OA, OB.

Proof. ∵ arc AEB subtends ∟ AOB at the centre and ∟ ACB at a pt. on the remaining part of the circumference.

∴ ∟ ACB = ½ ∟ AOB . . . . . . Th. 42.

Similarly ∟ ADB = ½ ∟ AOB . . . . . . Th. 42.

∴ ∟ ACB = ∟ ADB. Q.E.D.

Exercises.

882. Two ⊙s intersect at A and B; st. lines CAD and EAF are drawn cutting the ⊙s in C, D, E, F; prove that △s BCD, BEF are equiang.

883. PQ, RS, *chords of a* $\odot$, *intersect at* T ; *prove that* $\triangle$*s* PTS, QTR *are equiang.*

884. PQ, RS, chords of a $\odot$, when produced intersect at T ; prove that $\triangle$s PTS, QTR are equiang.

885. The $\angle$ subtended by the chord of a segment at any pt. within it $>$ the $\angle$ of the segment.

886. On the same st. line and on the same side of it there cannot be 2 different segments of $\odot$s containing $=$ $\angle$s. (Euc. III. 23.)

887. If 2 segments of $\odot$s on $=$ st. lines contain $=$ $\angle$s they are equal to one another. (Euc. III. 24.)

888. PQ is the chord of a segment and R is a pt. on the arc ; prove that $\angle$ RPQ $+$ $\angle$ RQP $=$ a const. for all positions of R.

889. P, Q *are the pts. of intersection of 2 fixed* $\odot$*s ; prove that any st. line through* P *and terminated by the* $\odot$*s subtends a const.* $\angle$ *at* Q. (Mad. Mat.)

* 890. PRQ, PSQ are 2 segments of $\odot$s having a common chord PQ and R is any pt. on the arc of the segment PRQ ; PR, QR are drawn and produced to meet the segment PSQ in T, S ; prove that the arc ST is of const. length for all positions of R.

* 891. *If the bisectors of the* $\angle$*s of a* $\triangle$ ABC *meet the circum-*$\odot$ *in* D, E, F, *prove that the* $\angle$*s of the* $\triangle$ DEF *are* $90^\circ - \frac{A}{2}$, $90^\circ - \frac{B}{2}$, $90^\circ - \frac{C}{2}$.

* 892. PQ and PS are chords of a $\odot$; R is the mid. pt. of PQ and T the mid. pt. of PS ; prove that $\angle$ PTR is const. for all positions of S.

* 893. *Two* $\odot$*s intersect at* P, Q ; *st. lines* RPS, RQT *are drawn from any pt.* R *on one* $\odot$ *; prove that they intercept an arc of const. length on the other* $\odot$.

* 894. PQ, RS, chords of a $\odot$, cut at rt. $\angle$s at T ; prove that the median of the $\triangle$ PTR through T is $\perp$ to QS.

* 895. A $\odot$ AOB, passing through the centre O of another $\odot$, cuts the latter $\odot$ at A and B ; a st. line APQ is drawn meeting the $\odot$ AOB in P and the other $\odot$ in Q ; prove that PB $=$ PQ. (Mad. Mat.)

* 896. I *is the in-centre of a* $\triangle$ PQR *and* PI *is produced to meet the circum-*$\odot$ *at* S ; *prove that* SQ $=$ SR $=$ SI. (Bom. Sch. Fin.)

* 897. O *is the orthocentre of a* $\triangle$ ABC *and* AO *is produced to meet* BC *at* D *and the circum-*$\odot$ *at* E ; *prove that* OD $=$ DE. (Bom. Mat.)

Theorem 44.

Gen. Enun. *If the line joining two points subtends equal angles at two other points on the same side of it, the four points are concyclic.*

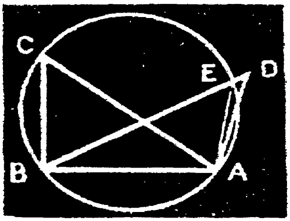


Fig. 267.

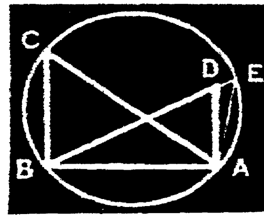


Fig. 268.

Part. Enun. Let the line joining A and B subtend = ∟s at C and D on the same side of AB.

It is reqd. to prove that

A, B, C, D *are concyclic.*

Const. Suppose a ⊙ drawn through A, B, C.
If this ⊙ does not pass through D, it must cut BD (or BD produced) in some other pt. E.
Join AE.

Proof. ∵ ∟s BCA, BEA are in the same segment.
∴ ∟ BCA = ∟ BEA . . . . . Th. 43.
But ∟ BCA = ∟ BDA . . . . . Hyp.
∴ ∟ BDA = ∟ BEA.
That is, an ext. ∟ of △ AED = an int. and opp. ∟, which is impossible . . . Th. 8, Cor. 4.
Hence ⊙ ABC must pass through D.
∴ A, B, C, D are concyclic. Q.E.D.

Exercises.

898. *If* 2 △*s on the same base and on the same side of it have = vert.* ∟*s the vertices of the* △*s and the ends of the base are concyclic.*

899. If 2 △s on the same base and on opp. sides of it have = vert. ∟s, their circum-radii are =.

900. *Find the locus of the vertex of a* △ *on a given base and of a given vert.* ∟. (Mad. Mat.)

901. ABC is a △ and D, E are the feet of the ⊥s from B, C on AC, AB respy.; prove that B, C, D, E are concyclic.

902. PQ, SR are ∥ chords of a ⊙ PQRS whose centre is O; if PR, QS intersect at T within the ⊙, prove that T, S, P, O are concyclic.

903. The internal bisectors of the vert. ∠s of △s on the same base and on the same side of it and having = vert. ∠s are concurrent.

904. PQ is the chord of a segment and R is any pt. on the arc: find the locus of the intersection of the bisectors of the ∠s RPQ, RQP.

905. PQ is the chord of a segment and R is any pt. on the arc; RP, RQ are produced to S, T respy.; find the locus of the intersection of the bisectors of the ∠s SPQ, TQP.

906. PQ is the chord of a segment and R is any pt. on the arc; find the locus of the pt. S on PR such that RS = RQ.

* 907. *Find the locus of the orthocentre of a △ on a given base and of a given vert. ∠.* (Mad. F. A.)

* 908. *Find the locus of the in-centre of a △ on a given base and of a given vert. ∠.* (Cal. Mat.)

* 909. *Find the locus of an e-centre of a △ on a given base and of a given vert. ∠.*

* 910. *Find the locus of the centroid of a △ on a given base and of a given vert. ∠.*

* 911. R, S are moveable pts. on the arc of a segment PRSQ, but the chord RS is always of const. length; find the locus of the intersection of PS and QR.

THEOREM 45.

(Euc. III. 31.)

Gen. Enun. (A) *The angle in a semicircle is a right angle.*
(B) *The angle in a major segment is acute.*
(C) *The angle in a minor segment is obtuse.*

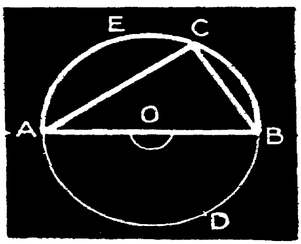


FIG. 269.

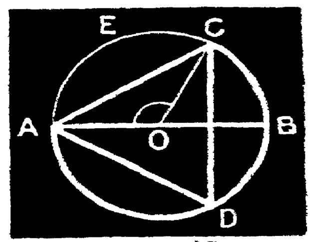


FIG. 270.

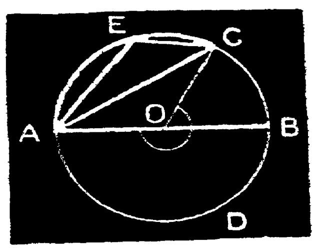


FIG. 271.

Part. Enun. Let AB be a diam. of a ⊙ whose centre is O, and let AC be a chord dividing the ⊙ into a major and minor segment.

Let D be any pt. on the arc of the major segment and E any pt. on the arc of the minor segment.

It is reqd. to prove that

(A) ∟ ACB is a rt. ∟ (see Fig. 269).
(B) ∟ ADC is acute (see Fig. 270).
(C) ∟ AEC is obtuse (see Fig. 271).

Proof of (A). ∟ ACB at the circumference = ½ straight ∟ AOB at the centre subtended by the same arc . . . . Th. 42.

∴ ∟ ACB is a rt. ∟. Q.E.D.

Proof of (B). Join OC (Fig. 270).

∟ADC at the circumference = ½ ∟AOC at the centre subtended by the same arc . . . Th. 42.

But ∟AOC < straight ∟AOB.

∴ ∟ADC is acute. Q.E.D.

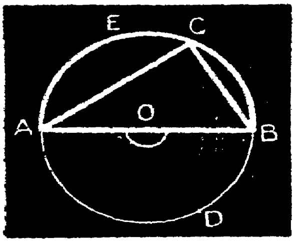


FIG. 269.

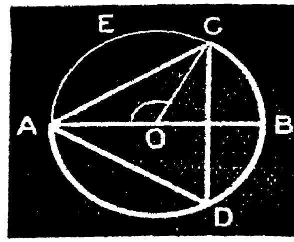


FIG. 270.

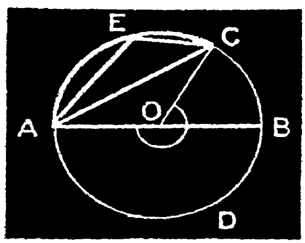


FIG. 271.

Proof of (C). Join OC (Fig. 271).

∟AEC at the circumference = ½ reflex ∟AOC at the centre subtended by the same arc . . . . . . Th. 42.

∴ ∟AEC is obtuse. Q.E.D.

Exercises.

912. ABC *is a* △ *rt.* ∟*d at* B; *prove that the* ⊙ *descd. on* AC *as diam. passes through* B.

913. If ⊙s are descd. on the sides of an isos. △ as diams. they will meet at the mid. pt. of the base.

914. If ⊙s are descd. on the sides of any △ as diams. they will meet, 2 and 2, on the sides of the △.

915. If ⊙s are descd. on the sides of a rhombus as diams. they will meet at the intersection of the diags.

916. Find the locus of the intersection of the diags. of a rhombus on a given base and on either side of it.

917. Two ⊙s intersect at P, Q, and through P diams. are drawn meeting the ⊙s again at R, S; prove that R, Q, S are collinear.

918. *AB is the diam of a ⊙ which intersects at* C *and* D *any ⊙ having* A *for centre; prove that* BC, BD *are tangents to the second ⊙.* (Mad. Mat.)

919. ABCD is a quadl. inscribed in a ⊙; AB produced meets DC produced at E; AD produced meets BC produced at F; if AC is a diam. of the ⊙, prove that B, D, F, E are concyclic.

920. *Find the locus of the pts. of contact of tangents drawn from the same pt. to concentric ⊙s.*

921. *A st. line of const. length slides with its ends along the arms of a rt. ∠; find the locus of the mid. pt.*

* 922. Two chords of a ⊙ intersect at rt. ∠s; prove that the sum of the opp. arcs which they intercept is $\frac{1}{2}$ the circumference.

* 923. State and prove the converse of Ex. 922.

* 924. Two chords of a ⊙ intersect at rt. ∠s; prove that each pair of opp. arcs which they intercept subtend supplementary ∠s at the centre.

* 925. PQR is a △ rt. ∠d at Q and on PQ as diam. a ⊙ is descd. cutting PR at S; prove that the tangent to the ⊙ at S bisects QR.

* 926. P, Q are the pts. of contact of 2 ∥ tangents to a ⊙ and PR, QS are drawn ∥ to one another and cut the ⊙ in T, X respy.; prove that PTQX is a rect.

* 927. *Find the locus of mid. pts. of chords of a ⊙ through a fixed pt.* (1) *inside the ⊙,* (2) *outside the ⊙.*

* 928. R is a pt. on a minor arc PQ of a ⊙ whose centre is O and RS is drawn ⊥ to the chord PQ; prove that ∠ PRS = ∠ ORQ.

* 929. Two ⊙s touch at O and st. lines AOB, COD are drawn at rt. ∠s to one another meeting one ⊙ in A, C and the other in B, D; if the line of centres meets the ⊙s again in E, F, prove that $EF^2 = AB^2 + CD^2$.

* 930. *Of all rt. ∠d △s having a given hypot., the perim. is greatest when it is isos.*

* 931. *Of all △s having a given base and vert. ∠, the perim. is greatest when it is isos.*

* 932. *Of all △s whose vertices lie on a given ⊙, the perim. is greatest when it is equilat.*

Def. 65. A cyclic quadrilateral is one whose vertices lie on the same circle.

THEOREM 46.

(Euc. III. 22.)

Gen. Enun. *The opposite angles of any cyclic quadrilateral are supplementary.*

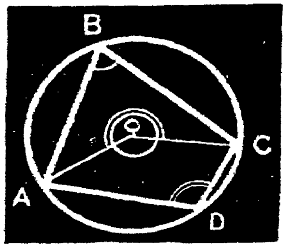


FIG. 272.

Part. Enun. Let ABCD be a quadl. inscd. in a ⊙ whose centre is O.

It is reqd. to prove that

(i) $\angle B + \angle D = 2$ *rt.* $\angle$s.
(ii) $\angle A + \angle C = 2$ *rt.* $\angle$s.

Const. Join OA, OC.

Proof. $\angle B$ at the circumference $= \frac{1}{2} \angle AOC$ at the centre subtended by the same arc . . . Th. 42.

Also $\angle D$ at the circumference $= \frac{1}{2}$ reflex $\angle AOC$ at the centre subtended by the same arc Th. 42.

$\therefore \angle B + \angle D = \frac{1}{2}(\angle AOC + \text{reflex} \angle AOC)$.

But $\angle AOC + \text{reflex} \angle AOC = 4$ rt. $\angle$s.

$\therefore \angle B + \angle D = 2$ rt. $\angle$s.

Similarly by joining OB, OD it can be proved that

$\angle A + \angle C = 2$ rt. $\angle$s. Q.E.D.

Exercises.

933. *The 2 segments into which a ⊙ is divided by any chord contain supplementary* $\angle$*s.*

934. *If the side* QR *of a cyclic quadl.* PQRS *is produced to* T, *the ext.* $\angle$ SRT = *the int. and opp.* $\angle$ SPQ.

935. PS, QR, the opp. sides of a cyclic quadl. PQRS, are produced to meet in T; prove that △s PQT, RST are equiang.

936. PS, QR, the opp. sides of a cyclic quadl. PQRS, are produced to meet in T; if PT = QT, prove that PQ is ∥ to SR.

937. *If a* ▱*gm is cyclic it must be a rect.* (Bom. Mat.)

938. If a ▱gm is cyclic its diags. intersect at the centre of its circum-⊙.

939. ABCD is a cyclic quadl. and ∠ B = ∠ D; prove that AC is a diam. of the circum-⊙.

940. The 3 alt. ∠s of any hexagon inscd. in a ⊙ together = 4 rt. ∠s.

941. ABCDEF is a hexagon inscd. in a ⊙ having AB ∥ to DE and BC ∥ to EF; prove that CD is ∥ to FA.

942. *Through* P, Q, *the pts. of intersection of* 2 ⊙*s, st. lines* APB, CQD *are drawn meeting one* ⊙ *in* A, C *and the other in* B, D; *prove that* AC *is* ∥ *to* BD.

943. ABCD is a cyclic quadl. and BC is produced to E; prove that the bisector of ∠ BAD meets the bisector of ∠ DCE on the circum-⊙. (Bom. Mat.)

944. ABCD is a cyclic quadl.; AB produced meets DC produced at F; AD produced meets BC produced at E; prove that ∠ ABC + ∠ ADC = ∠ AFD + ∠ AEB + 2 ∠ BAD.

* 945. *If a st. line is equally inclined to one pair of opp. sides of a cyclic quadl. it is also equally inclined to the other pair.* (Mad. Mat.)

* 946. If a st. line is equally inclined to one pair of opp. sides of a cyclic quadl. it is also equally inclined to the diags.

* 947. If a st. line is equally inclined to one pair of opp. sides of a cyclic quadl. the ⊥ upon it from the intersection of the diags. bisects the ∠ between the diags.

* 948. *The ext. bisector of the vert.* ∠ *of a* △ *meets the circum-*⊙ *in a pt. equidistant from the ends of the base.*

* 949. *The st. line joining the vertex of a* △ *to a pt. on the circum-*⊙ *equidistant from the ends of the base is either the int. or ext. bisector of the vert.* ∠.

* 950. Through one of the pts. of intersection of 2 ⊙s 2 st. lines AB, CD are drawn meeting the ⊙s again in A, B, C, D; prove that AC, BD make a const. ∠ with each other.

* 951. Two = ⊙s intersect in A, B and any st. line BCD is drawn to cut both ⊙s in C, D; prove that AC = AD. (Bom. Sch. Fin.)

* 952. ABCD is a quadl. having AB = AC = AD; prove that ∠ CBD + ∠ CDB = $\frac{1}{2}$ ∠ BAD.

Theorem 47.

Gen. Enun. *If a pair of opposite angles of a quadrilateral are supplementary, its vertices are concyclic.*

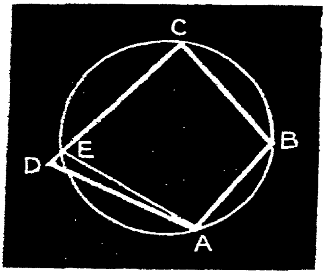


Fig. 273.

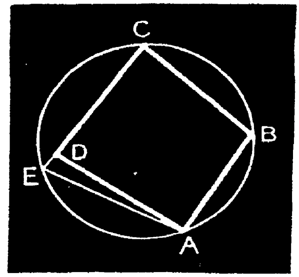


Fig. 274.

Part. Enun. Let ABCD be a quadl. whose opp. ∟s ABC and ADC are supplementary.

It is reqd. to prove that
the vertices A, B, C, D *are concyclic.*

Const. Suppose a ⊙ drawn through A, B and C.
If this ⊙ does not pass through D, it must cut CD (or CD produced) in some other pt. E.
Join AE.

Proof. ∵ ABCE is a cyclic quadl.
∴ ∟ ABC + ∟ AEC = 2 rt. ∟s . . Th. 46.
But ∟ ABC + ∟ ADC = 2 rt. ∟s . . Hyp.
∴ ∟ ABC + ∟ AEC = ∟ ABC + ∟ ADC.
∴ ∟ AEC = ∟ ADC.
That is, an ext. ∟ of △ AED = an int. and opp. ∟, which is impossible . . . Th. 8, Cor. 4.
Hence ⊙ ABC must pass through D.
∴ A, B, C, D are concyclic. Q.E.D.

Exercises.

953. *All rects. are cyclic quadls.*

954. *All isos. trapeziums are cyclic quadls.*

955. *If a st. line* AB *subtends supplementary* ∟*s at pts.* C, D *on opp. sides of it, prove that* A, B, C, D *are concyclic.*

956. ABCD is a cyclic quadl. and a st. line is drawn ∥ to BC meeting AB, CD in E, F respy.; prove that A, E, F, D are concyclic.

957. **O** is the orthocentre of a △ **ABC** and **D, E, F** are the feet of the ⊥s from **A, B, C** upon the opp. sides respy.; prove that **AEOF, BFOD, CDOE** are cyclic quadls.

958. **PQ** and **PR**, the = sides of an isos. △, are produced to **S** and **T** so that **QS** = **RT**; prove that **QRTS** is a cyclic quadl.

959. If **I** is the in-centre of a △ **ABC** and I_1 the centre of the e-⊙ touching **BC**, then **B, I, C** and I_1 are concyclic.

960. The int. bisectors of the ∠s of a convex quadl. form a cyclic quadl.

961. **PQ, RS** are ∥ chords of a ⊙ whose centre is **O**; **PR, QS** when produced meet at **T**; prove that **P, T, S, O** are concyclic.

* 962. *If in 2 ⊙s = chords subtend equal or supplementary ∠s at the circumferences the ⊙s must be =.*

* 963. If **O** is the orthocentre of a △ **ABC** the circum-⊙s of the △s **ABC, BOC, COA, AOB** are all =.

* 964. **ABCD** is a cyclic quadl.; **AB** produced meets **DC** produced at **F**; **AD** produced meets **BC** produced at **E**; prove that the circum-⊙s of △s **BCF, CDE** meet on the st. line joining **E** and **F**. (Bom. Sch. Fin.)

* 965. **T** is any pt. on the diag. **PR** of a ▱gm **PQRS**; prove that the circum-⊙s of △s **PST, RQT** meet on the diag. **QS**.

* 966. **D, E, F** *are the feet of the* ⊥*s from the vertices* **A, B, C** *of a* △ **ABC** *on the opp. sides; prove that the sides of the* △ **DEF** *are equally inclined to the sides of the* △ **ABC**. (Punj. Mat.)

The △ **DEF** is called the **Pedal Triangle** of the △ **ABC**.

* 967. In an acute ∠d △ each of the ⊥s from the vertices opp. sides is the int. bisector of an ∠ of the pedal △. (Cal. Mat.)

* 968. *If* I_1, I_2, I_3 *are the e-centres of a* △ **ABC**, *prove that* **ABC** *is the pedal* △ *of the* △ $I_1I_2I_3$.

* 969. *The feet of the* ⊥*s on the sides of a* △ *from any pt. on the circum-*⊙ *are in the same st. line.* (Cal. Mat.)

This st. line is called the **Pedal Line** or the **Simson Line** of the △ with respect to the pt.

* 970. **PQRS** is a quadl. whose sides pass through the vertices of another quadl. **ABCD**; every 2 adj. sides of **PQRS** are equally inclined to that side of **ABCD** which meets them both; prove that **ABCD** is a cyclic quadl.

* 971. **ABCD** is a cyclic quadl.; **AB** produced meets **DC** produced at **E**; **AD** produced meets **BC** produced at **F**; if the circum-⊙ of the △ **ADE** meets the line joining **E** and **F** at **G**, prove that **CDFG, BCGE, ABGF** are cyclic quadls.

* 972. **P, Q, R** are any pts. on the sides **BC, CA, AB** respy. of a △ **ABC**; prove that the circum-⊙s of the △s **AQR, BRP, CPQ** are concurrent.

* 973. **S** is a pt. in the base **QR** of a △ **PQR**, and through **Q, S, R** lines are drawn ⊥ to **PQ, PS, PR** respy. meeting one another in **T, X, V**; prove that **P, T, X, V** are concyclic.

* 974. From any pt. R on an arc PRQ of a ⊙ whose centre is O, RS and RT are drawn ⊥ to the radii OP and OQ respy.; prove that ST is a const. length for all positions of R.

On the Nine-points Circle.

* 975. *In any △ the mid. pts. of the sides and the mid. pts. of the lines joining the orthocentre to the vertices lie on a ⊙.*

* 976. *In any △ the mid. pts. of the sides and the mid. pts. of the lines joining the orthocentre to the vertices and the feet of the ⊥s from the vertices to the opp. sides lie on a ⊙.* (Bom. Prev.)

This ⊙ is called the Nine-points Circle of the △.

* 977. *In any △ the nine-points ⊙ is the circum-⊙ of the pedal △.*

* 978. *The distance of the orthocentre from any vertex of a △ = twice the ⊥ from the circum-centre to the opp. side.*

* 979. *In any △ the nine-points radius = $\frac{1}{2}$ the circum-radius.*

* 980. *In any △ the nine-points centre lies midway between the orthocentre and the circum-centre.*

Theorem 48.

(Euc. III. 32.)

Gen. Enun. *If a straight line touches a circle and from the point of contact a chord is drawn, the angles which this chord makes with the tangent are respectively equal to the angles in the alternate segments.*

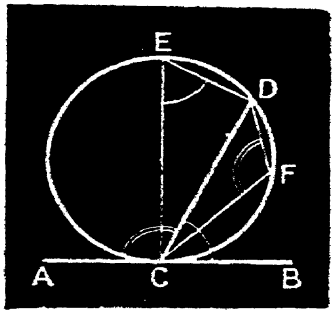


Fig. 275.

Part. Enun. Let the st. line AB touch the ⊙ DEF at the pt. C and let the chord CD be drawn.

It is reqd. to prove that:—

(i) $\angle$ BCD = *any* $\angle$ *in the alt. segment* DEC.

(ii) $\angle$ ACD = *any* $\angle$ *in the alt. segment* DFC.

Const. Through C draw the diam. CE.
Take any pt. F on the arc DFC.
Join CF, DF, DE.

Proof. (i) ∵ AB is a tangent and CE a diam.
∴ $\angle$ ECB is a rt. $\angle$ . . . . . Th. 40.
Again ∵ $\angle$ EDC, being in a semi-⊙, is a rt. $\angle$. Th. 45.
∴ $\angle$ DEC + $\angle$ ECD = a rt. $\angle$ . . . Th. 8.
∴ $\angle$ ECB = $\angle$ DEC + $\angle$ ECD.
From these equals take away the common $\angle$ ECD.
∴ $\angle$ BCD = $\angle$ DEC.
Hence $\angle$ BCD = any $\angle$ in the alt. segment DEC . . . . Th. 43.

(ii) ∵ DECF is a cyclic quadl.
∴ $\angle$ DFC = 180° − $\angle$ DEC . . . Th. 46.
= 180° − $\angle$ BCD . . Proved above.
= $\angle$ ACD . . . . Th. 1.
Hence $\angle$ ACD = any $\angle$ in the alt. segment DFC . . . . Th. 43.

Q.E.D.

Exercises.

981. *State and prove the converse of Theorem* 48. (Bom. Prev.)

982. Two tangents **AB**, **AC** are drawn to a ⊙; **D** is any pt. on the ⊙ outside △ **ABC**; show that ∟ **ABD** + ∟ **ACD** = a constant. (Punj. Mat.)

983. If one of 2 intersecting ⊙s passes through the centre of the other, the tangent to the first at one of the pts. of intersection and the common chord make = ∟s with the radius to that pt. from the centre of the second. (Cal. F. E.)

984. Two tangents **AP**, **AQ** are drawn to a ⊙ and **B** is the mid. pt. of the arc **PQ** convex to **A**; show that **PB** bisects ∟ **APQ**. (Allah. Mat.)

985. *If* 2 ⊙*s intersect, the* ∟*s subtended at the pts. of intersection by a common tangent are supplementary.* (Bomb. Mat.)

986. **AB** is a chord of a ⊙; the ⊥ from **A** on the tangent at **B** meets the st. line through **B** at rt. ∟s to **AB** in **C**; prove that **AC** = the diam. of the ⊙. (Bomb. Prev.)

987. If 2 ⊙s touch internally, any chord of the greater ⊙ which touches the less shall be divided at the pt. of its contact into segments which subtend = ∟s at the pt. of contact of the 2 ⊙s. (Bomb. Sch. Fin.)

988. *If* 2 ⊙*s touch internally, and a st. line is drawn to cut them, the segments of it intercepted between the* ⊙*s subtend* = ∟*s at the pt. of contact.* (Bomb. Sch. Fin.)

* 989. *Prove Th.* 48 *by the Method of Limits.*

* 990. Two ⊙s touch internally at **O**; **ABCD** is a chord cutting the outer ⊙ in **A** and **D** and the inner ⊙ in **B** and **C**; **OA**, **OD** cut the inner ⊙ in **A′**, **D′** respy. and **OB**, **OC** produced cut the outer ⊙ in **B′**, **C′** respy.; prove that **B′C′** is || to **A′D′**. (Mad. Mat.)

* 991. **C** is the centre of a ⊙; **CA**, **CB** are two radii at rt. ∟s to each other: from **B** any chord **BP** is drawn cutting **CA** at **N**; show that **BA** touches the circum-⊙ of the △ **ANP**. (Bomb. Mat.)

* 992. **A** is a pt. of intersection of 2 ⊙s; through **A** any 2 st. lines **BAC**, **DAE** are drawn terminated by the ⊙s at **B**, **C** and **D**, **E**; prove that the chords **BD**, **CE** intersect at the same ∟ as the tangents to the ⊙s at the pt. **A**. (Mad. Mat.)

* 993. The diam. **BA** of a ⊙ is produced to **P** so that **AP** = the radius; through **A** the tangent **AED** is drawn and from **P** the tangent **PEC** touching the ⊙ at **C** and meeting **AED** at **E**; **BC** is joined and produced to meet **AED** at **D**; prove that **DEC** is an equilat. △. (Cal. F. E.)

* 994. **P** is a pt. in **APB** an arc of a ⊙, such that arc **AP** = twice arc **PB**; the tangent at **P** meets the chord **AB** produced at **R** and **AQ** ⊥ to **AB** at **Q**; prove that **QP** = **PR**. (Bomb. Mat.)

* 995. *Two* ⊙*s touch externally; a st. line cuts one of them in* **A**, **D** *and the other in* **B**, **C**; *show that* **AB** *and* **CD** *subtend supplementary* ∟*s at the pt. of contact.* (Mad. Mat.)

* 996. A △ PQR is inscd. in a ⊙ whose centre is A and the ⊥ PN on QR touches the ⊙ at P; if PB ⊥ to PQ meets QR in B, prove that BN = NR. (Bomb. Prev.)

* 997. From any pt. A on a given ⊙ 2 chords AB, AD are drawn touching another given ⊙ whose centre C is on the former ⊙; prove that BD is || to the tangent at C. (Mad. Mat.)

* 998. *If 2 ⊙s intersect, their common tangents subtend at one of the pts. of intersection ∠s which are supplementary.* (Bomb. Prev.)

PRACTICAL SECTION.

EQUIVALENT AREAS.

PROBLEM 13.

Gen. Enun. *To construct a triangle equivalent to a given quadrilateral.*

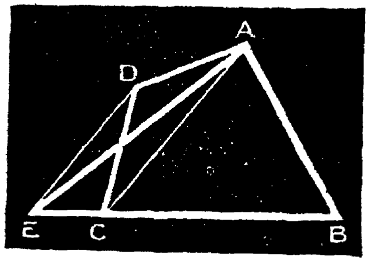


FIG. 276.

Part. Enun. Let ABCD be a given quadl.
It is reqd. to construct a △ *equivalent to the quadl.* ABCD.

Const. Join AC.
Through D draw DE ‖ to AC meeting BC produced in E . . . . Prob. 6.
Join AE.
Then shall △ ABE = quadl. ABCD.

Proof. ∵ △ ACE = △ ACD, being on the same base AC and of the same altitude . . Th. 28, Cor. 2.
∴ △ ABC + △ ACE = △ ABC + △ ACD.
That is, △ ABE = quadl. ABCD. Q.E.F.

Cor. *To construct a triangle equivalent to a given polygon.*

Exercises.

999. Divide a given ▱gm into 7 equiv. ▱gms in 2 different ways.

1000. Divide a given △ into 5 equiv. △s in 3 different ways.

1001. On the base AB of a rect. ABCD construct an equiv. ▱gm having a given ∟.

1002. On the base AB of a rect. ABCD construct an equiv. ▱gm having a side of given length > AD.

1003. On the base AB of a rect. ABCD construct an equiv. ▱gm having its perim. of given length. When is this imposs.?

1004. Construct a rhombus equiv. to a given rect. (Mad. Mat.)

* 1005. Construct a ▱gm having adj. sides of given lengths and equiv. to a given rect. When is this imposs.?

1006. Construct an isos. △ equiv. to a given △ and standing on the same base. (Allah. Mat.)

1007. On the base AB of a △ ABC construct an equiv. △ having a given ∟.

1008. On the base AB of a △ ABC construct an equiv. rt. ∟d △.

1009. On the base AB of a △ ABC construct an equiv. △ having a side of given length. When is this imposs.?

1010. On the base AB of a △ ABC construct an equiv. △ having its vertex on a given st. line. When is this imposs.?

1011. On the base AB of a △ ABC construct an equiv. △ having its vertex equidistant from 2 given pts. When is this imposs.?

1012. Construct a ▱gm equiv. to a given △ and having a given ∟. (Euc. I. 42.)

* 1013. Given 2 sides and the area, construct the △. (Mad. F. A.)

* 1014. Construct a ▱gm equiv. to a given △ and having the same perim. (Mad. Mat.)

* 1015. Construct a ▱gm equiv. to a given ▱gm and having its diags. of given lengths.

* 1016. From a given pt. in one of the sides of a △ (or in a side produced) draw a st. line to meet the other side (produced if necessary) so that the △ thus formed shall be equiv. to the given △.

* 1017. Construct a △ on a given base and equiv. to a given △.

* 1018. Construct a △ of given altitude and equiv. to a given △. (Punj. F. E.)

* 1019. Construct a △ equiv. to the sum of 2 given △s.

* 1020. Construct a △ equiv. to the difference of 2 given △s.

* 1021. Construct a △ of given area having its base along a given st. line and its vertex at a given pt.

* 1022. Bisect a ▱gm by a st. line through a given pt. in one of its sides.

* 1023. Bisect a ▱gm by a st. line through a given pt. in one of its sides produced.

* 1024. Bisect a △ by a st. line through a given pt. in a side. (Bom. Sch. Fin.)

* 1025. Trisect a △ by st. lines through a given pt. in a side.

* 1026. Cut off a fourth part of a △ by a st. line through a given pt. in a side.

* 1027. Cut off an *n*th part of a △ by a st. line through a given pt. in a side.

* 1028. Bisect a quadl. by a st. line through an angular pt. (Cal Mat.)

* 1029. Cut off a third part of a quadl. by a st. line through an angular pt.

* 1030. Cut off an nth part of a quadl. by a st. line through an angular pt.

* 1031. Divide a △ into n = parts by st. lines through a given pt. in a side.

* 1032. PQR is a △; S is a pt. in PQ; find a pt. T in QR such that △ PST = △ PRT.

* 1033. Divide a △ into 3 equal parts by st. lines drawn to the vertices from a pt. within the △.

* 1034. Construct a ▱gm equiv. to a given ▱gm and having a side = a given st. line.

* 1035. To a given st. line apply a ▱gm equiv. to a given △ and having a given ∟. (Euc. I. 44.)

* 1036. Construct a ▱gm equiv. to a given rectil. fig. and having a given ∟. (Euc. I. 45.)

* 1037. Given 2 sqs., construct a sq. equiv. to their sum.

* 1038. Given 3 sqs., construct a sq. equiv. to their sum.

* 1039. Given 2 sqs., construct a sq. equiv. to their difference.

* 1040. Divide a given st. line into 2 parts such that the sum of their sqs. = a given sq. (Allah. Mat.)

* 1041. Construct a sq. equiv. to 3 times a given sq. (Mad. Mat.)

* 1042. Find a pt. in the diag. of a sq. produced from which, if a st. line is drawn ∥ to any side of the sq. and meeting another side produced, it will form, together with the produced diag. and the produced side, a △ = the sq. (Bom. Sch. Fin.)

* 1043. In a given st. line AB find a pt. D such that $AD^2 - BD^2$ = a given sq. whose side is not > AB. (Bom. Mat.)

* 1044. Construct a sq. = $\frac{1}{2}$ of a given sq.

* 1045. Cut off the corners of a given sq. so that it may become a reg. octagon.

* 1046. A and B are 2 inaccessible pts. on the far bank of a river; give a construction for finding the distance between them.

* 1047. Find a pt. P in AB such that $AP^2 = 2PB^2$. (Bom. Sch. Fin.)

* 1048. Construct a sq. whose area is twice, three times, four times . . . a given sq.

* 1049. Draw st. lines measuring 1, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$ . . . in.

CIRCLES.

PROBLEM 14.

(Euc. III. 1.)

Gen. Enun. *To find the centre of a circle of which an arc is given.*

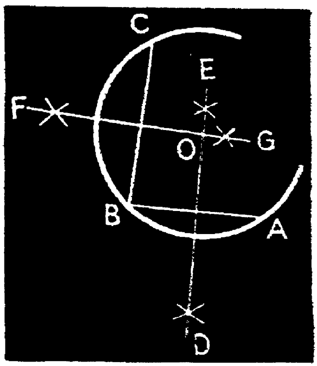


FIG. 277.

Part. Enun. Let ABC be a given arc of a ⊙.
It is reqd. to find the centre of the ⊙ of which ABC *is an arc.*

Const. Draw 2 chords AB, BC.
Draw DE the perp. bisector of AB . . Prob. 2.
Draw FG the perp. bisector of BC . . Prob. 2.
Let DE and FG meet at O . . . . (see Ex. 44.)
Then shall O be the centre of the ⊙ of which ABC is an arc.

Proof. ∵ DE is the perp. bisector of the chord AB.
∴ DE passes through the centre of the ⊙ of which ABC is an arc . . . . . Th. 35, Cor.
Similarly FG passes through the centre.
∴ O, the only pt. common to DE and FG, must be the centre. Q.E.F.

Problem 15.

(Euc. III. 30.)

Gen. Enun. *To bisect a given arc of a circle.*

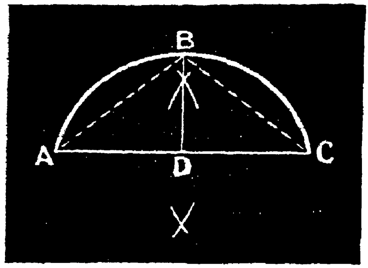


Fig. 278.

Part. Enun. Let ABC be a given arc of a ⊙.
It is reqd. to bisect the arc ABC.

Const. Join AC and draw BD, bisecting AC at rt. ∟s and meeting the arc ABC in B . . . Prob. 2.
Then shall the arc ABC be bisected at B.

Proof. Join BA, BC.
∵ B is equidistant from A and C . . . Th. 24.
∴ chord AB = chord BC.
∴ minor arc AB = minor arc BC . . . Th. 38.
∴ arc ABC is bisected at B. Q.E.F.

Exercises.

1050. An arc of a ⊙ being given, desc. the whole ⊙ of which it is an arc. (Euc. III. 25.)

1051. Two chords of a ⊙ being given in position and magnitude, desc. the ⊙.

1052. Desc. a ⊙ to pass through 2 given pts. and having its centre on a given st. line. When is this imposs.? (Bom. Sch. Fin.)

1053. Desc. a ⊙ to pass through 2 given pts. and having its radius of given length. When is this imposs.?

1054. Desc. a ⊙ with a given centre to cut the circumference of a given ⊙ in 2 = parts.

1055. In a given ⊙ draw a chord whose distance from the centre = ½ its length.

* 1056. Through a given pt. within a ⊙ draw the shortest chord.

* 1057. Through a given pt. within a ⊙ draw a chord which shall be bisected at that pt.

* 1058. Desc. 2 ⊙s of given radii having their common chord of given length.

* 1059. Through R, one of the pts. of intersection of 2 ⊙s, draw a st. line meeting the ⊙s in P and Q such that PR = RQ.

TANGENCY.

PROBLEM 16.

(Euc. III. 17.)

Gen. Enun. *To draw a tangent to a circle :—*
(A) *At a given point on the circumference.*
(B) *From a given point outside the circumference.*

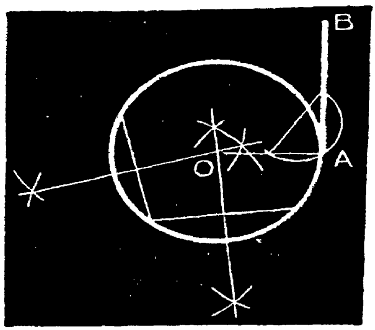


FIG. 279.

Part. Enun. (A). Let A be a pt. on the circumference of a $\odot$ (Fig. 279).

It is reqd. to draw a tangent to the $\odot$ at A.

Const. Find O the centre of the $\odot$ . . . Prob. 14.
Join OA.
Draw AB $\perp$ to OA . . . . . Prob. 3.
Then shall AB be a tangent to the $\odot$ at A.

Proof. $\because$ AB is $\perp$ to radius OA.
$\therefore$ AB is a tangent to the $\odot$ at A . Th. 40, Cor. 2.

Q.E.F.

Part. Enun. (B). Let A be a pt. outside the circumference of a $\odot$ (Fig. 280).

It is reqd. to draw a tangent to the $\odot$ from A.

Const. Find O the centre of the $\odot$ . . . Prob. 14.
Join OA.
On OA as diam. desc. a $\odot$.
This $\odot$ will cut the given $\odot$ in two pts. since O is within and A is without the given $\odot$, and in two pts. only.
Let B and C be these pts.
Join AB, AC.
Then shall AB, AC be tangents to the $\odot$ from A.

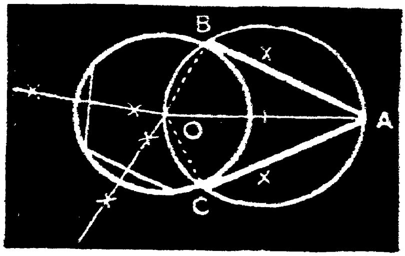


FIG. 280.

Proof. Join OB, OC.

∵ segment OBA is a semi-⊙.

∴ ∠ OBA is a rt. ∠ . . . . . Th. 45.

∴ AB is ⊥ to radius OB.

∴ AB is a tangent to the ⊙ from A . . Th. 40, Cor. 2.

Similarly AC is a tangent to the ⊙ from A. Q.E.F.

Moreover any other st. line drawn from A to meet the given ⊙ at D, say, is not a tangent since (as can be easily proved), ∠ ADO > a rt. ∠ or < a rt. ∠ according as D is within or without the ⊙ ABC. Thus:

Two tangents, and two only, can be drawn to a circle from an external point.

Exercises.

1060. Desc. a ⊙ with a given centre to touch a given st. line.

1061. Desc. a ⊙ with a given centre to touch a given ⊙. How many solutions are there?

1062. Desc. a ⊙ of given radius to touch a given st. line at a given pt. How many solutions are there?

1063. Desc. a ⊙ of given radius to touch a given ⊙ at a given pt. How many solutions are there?

1064. Desc. several ⊙s of given radius to touch a given st. line.

1065. Desc. several ⊙s of given radius to touch a given ⊙.

1066. Draw a tangent to a given ⊙ ∥ to a given st. line.

1067. Draw a tangent to a given ⊙ ⊥ to a given st. line.

1068. Draw 2 tangents to a given ⊙ containing an ∠ = a given ∠.

1069. Draw a st. line through a given pt. at a given distance from a given pt.

1070. Desc. several ⊙s to touch 2 ∥ st. lines.

1071. Desc. several ⊙s to touch 2 concent. ⊙s.

1072. Draw 2 concent. ⊙s such that each chord of the outer that touches the inner = the diam. of the inner.

*1073. Through a given pt. within a ⊙ draw a chord of given length. When is this imposs.?

*1074. Through a given pt. without a ⊙ draw a st. line so that the part intercepted by the ⊙ is of given length.

*1075. Within a given ⊙ draw a chord of given length ∥ to a given st. line.

*1076. Within a given ⊙ draw a chord of given length ⊥ to a given st. line.

*1077. Desc. a ⊙ to pass through a given pt. and touch a given st. line at a given pt. (Allah. Mat.)

*1078. Desc. a ⊙ to pass through a given pt. and touch a given ⊙ at a given pt.

*1079. Desc. several ⊙s to touch 2 given intersecting st. lines.

*1080. Desc. a ⊙ of given radius to touch 2 given intersecting st. lines. How many solutions are there?

*1081. Desc. a ⊙ of given radius to touch a given ⊙ and a given st. line.

*1082. Desc. a ⊙ of given radius to touch 2 given ⊙s.

*1083. Desc. a ⊙ of given radius to touch a given st. line and pass through a given pt.

*1084. Desc. 3 ⊙s of given radii to touch each other externally.

*1085. Desc. 2 ⊙s of given radii to touch each other externally and to touch a third ⊙ of given radius internally.

*1086. Desc. 2 ⊙s of given radii to touch each other and a given st. line on the same side of it.

*1087. Find a pt. in a given st. line from which the tangent drawn to a given ⊙ is of given length.

*1088. Find a pt. from which the tangents drawn to 2 given ⊙s are of given lengths.

*1089. Desc. a ⊙ to touch a given ⊙ and a given st. line at a given pt. (Cal. Mat.)

*1090. Desc. a ⊙ to touch a given st. line and a given ⊙ at a given pt. (Mad. Mat.)

PROBLEM 17.

Gen. Enun. *To draw a* $\genfrac{}{}{0pt}{}{\textit{direct}}{\textit{transverse}}$ *common tangent to two circles.*

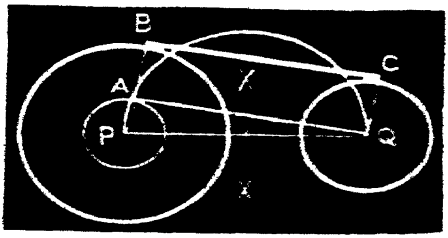


FIG. 281.

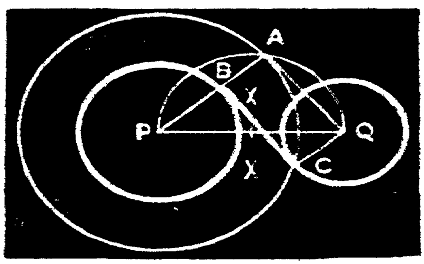


FIG. 282.

Part. Enun. Let **P** be the centre of the larger and **Q** the centre of the smaller of 2 given ⊙s.

It is reqd. to draw a $\genfrac{}{}{0pt}{}{\textit{direct}}{\textit{transverse}}$ *common tangent to the* ⊙ *whose centres are* **P** *and* **Q**.

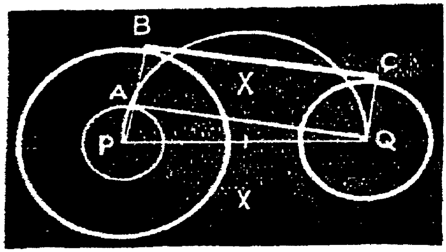


FIG. 281.

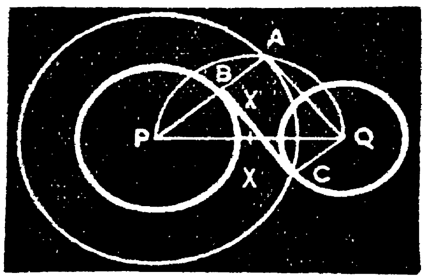


FIG 282.

Const. With centre P and radius = the $\begin{matrix}\text{difference}\\ \text{sum}\end{matrix}$ of the radii of the given ⊙s desc. a ⊙.

From Q draw a tangent QA to this ⊙ . . Prob. 16.

Join PA and let $\begin{matrix}\text{PA produced}\\ \text{PA}\end{matrix}$ meet the circumference of the larger ⊙ at B.

Draw QC ∥ to PB $\begin{matrix}\text{in the same sense}\\ \text{in the opposite sense}\end{matrix}$ . . Prob. 6.

Let QC meet the circumference of the smaller ⊙ at C.

Join BC.

Then shall BC be a $\begin{matrix}\text{direct}\\ \text{transverse}\end{matrix}$ common tangent to the ⊙s whose centres are P and Q.

Proof. ∵ AB and QC are = and ∥.

∴ BC is ∥ to AQ . . . . . Ex. 151.

∴ ABCQ is a ▱gm.

And $\angle$s PAQ, PBC are rt. $\angle$s . . . Ths. 40, 6.

$\therefore$ ABCQ is a rectangle.

$\therefore$ $\angle$ BCQ is a rt. $\angle$.

$\therefore$ BC is a $\frac{\text{direct}}{\text{transverse}}$ common tangent to

the $\odot$s whose centres are P and Q . . Th. 40, Cor. 2.

Q.E.F.

Exercises.

1091. Draw the direct common tangents to 2 = $\odot$s.

1092. Draw the transverse common tangents to 2 = $\odot$s.

1093. Draw the common tangents to 2 intersecting $\odot$s.

1094. Draw the common tangents to 2 $\odot$s touching each other externally.

1095. Draw a common tangent to 2 $\odot$s touching each other internally.

* 1096. Draw a st. line cutting 2 $\odot$s so that the intercepted chords may be of given lengths. (Mad. F. A.)

* 1097. Draw a st. line touching one of 2 $\odot$s and cutting the other so that the intercepted chord may be of given length.

CIRCUMSCRIBED, INSCRIBED AND ESCRIBED CIRCLES.

PROBLEM 18.

(Euc. IV. 5.)

Gen. Enun. *To circumscribe a circle about a given triangle.*

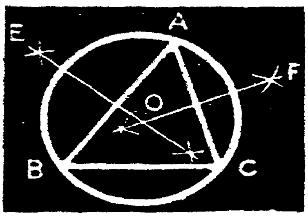


FIG. 283.

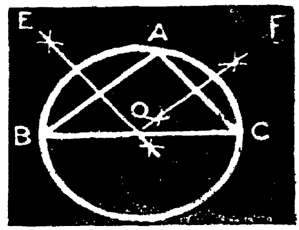


FIG. 284.

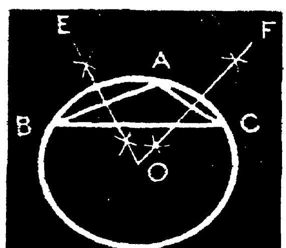


FIG. 285.

Part. Enun. Let ABC be a given △.
It is reqd to circumscribe a ⊙ about the △ ABC.

Const. Draw EO, FO the perp. bisectors of AB, AC respy. . . . . Prob. 2.
Let EO, FO meet at O (see Ex. 44).
With centre O and radius OA desc. a ⊙.
This shall be a ⊙ circumscribed about △ ABC.

Proof. ∵ every pt. on EO is equidistant from A and B Th. 24.
And every pt. on FO is equidistant from A and C Th. 24.
∴ O, the point common to EO and FO, is equidistant from A, B and C.
Hence the ⊙ descd. with centre O and radius OA will pass through A, B and C. Q.E.F.

Exercises.

1098. Circumsc. a ⊙ about a quadl. whose opp. ∠s are supplementary.

1099. Construct a △ of given base, given altitude and given circum-radius.

1100. Circumsc. a ⊙ about a given sq. (Euc. IV. 9.)

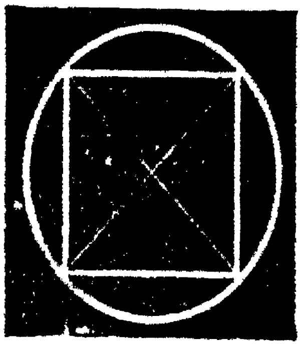


Fig. 285a.

1101. Circumsc. a ⊙ about a given reg. hexagon.

1102. Insc. a sq. in a given ⊙. (Euc. IV. 6.)

1103. Insc. a reg. octagon in a given ⊙.

* 1104. Insc. a reg. hexagon in a given ⊙. (Euc. IV. 15.)

* 1105. Insc. an equilat. △ in a given ⊙.

* 1106. Insc. a reg. dodecagon in a given ⊙.

Problem 19.

(Euc. IV. 4.)

Gen. Enun. *To inscribe a circle in a given triangle.*

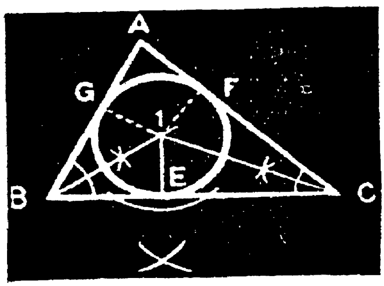


Fig. 286.

Part. Enun. Let ABC be a given △.
It is reqd. to inscribe a ⊙ in the △ ABC.

Const. Draw BI, CI the bisectors of ∟s ABC, ACB respy. . . . . Prob. 1.
Let BI, CI meet at I (see Ex. 65).
Draw IE ⊥ to BC . . . . . . Prob. 4.
With centre I and radius IE desc. a ⊙.
This shall be a ⊙ inscd. in the △ ABC.

Proof. Draw IF, IG ⊥s to CA, AB respy. . . . Prob. 4.
Now ∵ every pt. on BI is equidistant from BA, BC Th. 25.
And every pt. on CI is equidistant from CA, CB Th. 25.
∴ I, the pt. common to BI, CI is equidistant from AB, BC, CA.
That is, IE = IF = IG.
Hence the ⊙ descd. with centre I and radius IE will pass through E, F, G.
And this ⊙ will be touched by the sides of the △ at E, F, G, since the ∟s at E, F, G are rt. ∟s . . . . . . Th. 40, Cor. 2.

Q.E.F.

PROBLEM 20.

Gen. Enun. *To draw an escribed circle of a given triangle.*

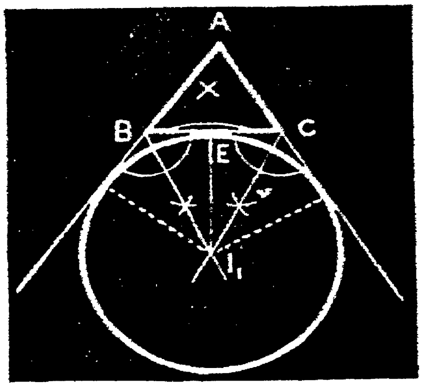


FIG. 287.

Part. Enun. Let ABC be a given $\triangle$.
It is reqd. to draw an escribed $\odot$ *of the* $\triangle$ ABC *to touch* AB *produced,* AC *produced and* BC.

Const. Draw BI_1, CI_1 the bisectors of the ext. $\angle$s at B and C . . . . Prob. 1.
Let BI_1, CI_1 meet at I_1 (see Ex. 85).
Draw $I_1E \perp$ to BC . . . . . . Prob. 4.
With centre I_1 and radius I_1E desc. a $\odot$.
This shall be an escribed $\odot$ of the $\triangle$ ABC.

Proof. As in Prob. 19. Q.E.F.

Exercises.

1107. Desc. a $\odot$ to cut the sides of a given $\triangle$ so that the intercepted arcs are =. (Punj. Mat.)

1108. Desc. 2 $\odot$s to touch 3 st. lines of which 2, and only 2, are $\parallel$.

1109. Desc. 3 $\odot$s with given centres to touch each other. How many solutions are there?

1110. Insc. a $\odot$ in a given sector of a $\odot$.

1111. Desc. a $\odot$ to touch a given $\odot$ and the tangents to the $\odot$ at 2 given pts.

1112. Insc. a ⊙ in a given sq. (Euc. IV. 8.)

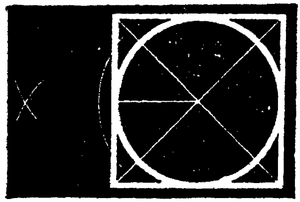


Fig. 287A.

1113. Insc. a ⊙ in a given rhombus.

1114. Insc a ⊙ in a given reg. hexagon.

1115. Circumsc. a sq. about a given ⊙. (Euc. IV. 7.)

* 1116. Circumsc. a reg. octagon about a given ⊙.

* 1117. Circumsc. a reg. hexagon about a given ⊙. (Punj. Mat.)

* 1118. Circumsc. a reg. dodecagon about a given ⊙.

* 1119. Circumsc. a rhombus having an ∟ = a given ∟ about a given ⊙.

* 1120. Construct a △ having given its 3 e-centres.

THE "ALTERNATE" SEGMENT.

PROBLEM 21.

(Euc. III. 33.)

Gen. Enun. *On a given finite straight line to describe a segment of a circle which shall contain an angle equal to a given angle.*

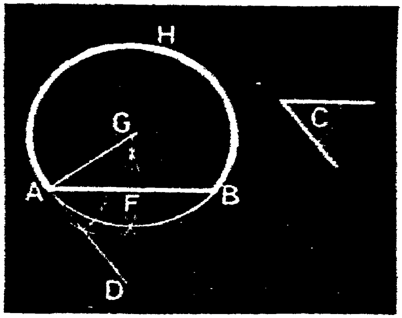


FIG. 288.

Part. Enun. Let AB be a given finite st. line and C a given ∟. *It is reqd. to desc. on AB a segment of a ⊙ containing an* ∟ = ∟ C.

Const.	At A make ∟ BAD = ∟ C	Prob. 5.
	Draw AG ⊥ to AD	Prob. 3.
	Draw FG, bisecting AB at rt. ∟s and meeting AG in G	Prob. 2.
	Then G is equidistant from A and B . . .	Th. 24.
	With centre G and radius GA desc. a ⊙ ABH.	
	Then the segment ABH shall contain an ∟ = ∟ C.	
Proof.	∵ AG is ⊥ to AD	Const.
	∴ AD touches the ⊙ ABH	Th. 40, Cor. 2.
	And AB is a chord of the ⊙ ABH.	
	∴ ∟ BAD = the ∟ in the alt. segment AHB	Th. 48.
	But ∟ BAD = ∟ C	Const.
	∴ the segment AHB contains an ∟ = ∟ C.	Q.E.F.

Cor. 1. *On a given finite straight line describe a segment of a circle which shall contain a right angle.*

Cor. 2. *From a given circle cut off a segment which shall contain an angle equal to a given angle.* (Euc. III. 34.)

Draw a tangent to the ⊙ and from the pt. of contact draw a chord making with the tangent an ∟ = the given ∟.

PROBLEM 22.

(Euc. IV. 2.)

Gen. Enun. *In a given circle to inscribe a triangle equiangular to a given triangle.*

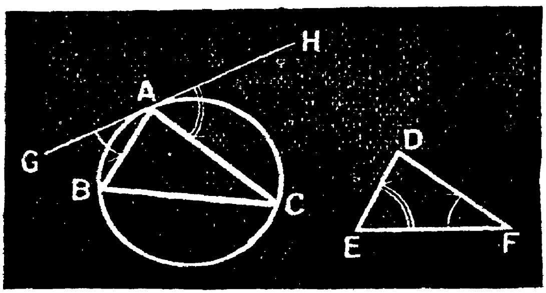


FIG. 289.

Part. Enun. Let ABC be a given ⊙ and DEF a given △.
It is reqd. to inscribe in the ⊙ ABC *a* △ *equiang. to the* △ DEF.

Const. At any pt. A on the ⊙ draw the tangent GAH Prob. 16.
Make ∟s GAB, HAC, on the same side of GAH, = ∟s DFE, DEF respy. . . Prob. 5.
Let AB, AC cut the ⊙ at B, C respy.
Join BC.
Then shall the △ ABC be inscd. in the ⊙ ABC and be equiang. to the △ DEF.

Proof. ∵ AB is a chord of the ⊙ drawn from the pt. of contact of the tangent AG.
∴ ∟ GAB = ∟ ACB in the alt. segment . Th. 48.
But ∟ GAB = ∟ DFE . . . . . Const.
∴ ∟ ACB = ∟ DFE.
Similarly ∟ ABC = ∟ DEF.
∴ third ∟ BAC = third ∟ EDF . . . . Th. 8.
∴ △ ABC, inscd. in ⊙ ABC, is equiang. to △ DEF. Q.E.F.

PROBLEM 23.

(Euc. IV. 3.)

Gen. Enun. *About a given circle to circumscribe a triangle equiangular to a given triangle.*

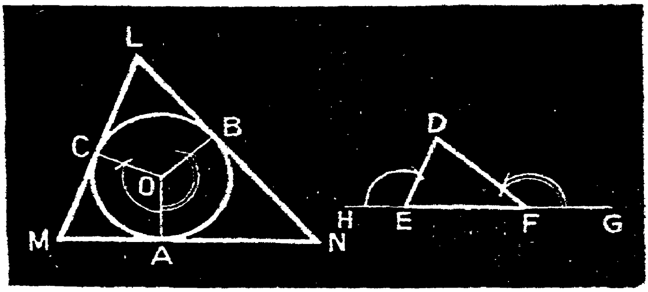


FIG. 290.

Part. Enun. Let ABC be a given ⊙ and DEF a given △.
It is reqd. to circumscribe about the ⊙ ABC *a* △ *equiang. to the* △ DEF.

Const. Produce EF both ways to G and H.
Find O the centre of the ⊙ . . . . . Prob. 14.
Draw any radius OA.
Make ∟s AOB, AOC, on opp. sides of AO, = ∟s DFG, DEH respy. . . . . . Prob. 5.
Through A, B, C draw MN, NL, LM ⊥s to OA, OB, OC . . . . Prob. 3.
Then shall the △ LMN be circumscd. about the ⊙ ABC and be equiang. to the △ DEF.

Proof. ∵ MN, NL, LM are ⊥s to radii of the ⊙ at their extremities . . . . Const.
∴ the △ LMN is circumscd. about the ⊙ Th. 40, Cor. 2.
Again ∵ the 4 ∟s of the quadl. AOBN together = 4 rt. ∟s . . . . . . Th. 8.
And, of these, ∟s OAN, OBN are rt. ∟s . Const.
∴ ∟s ANB, AOB together = 2 rt. ∟s.
= ∟ DFE + ∟ DFG Th. 1.
But ∟ AOB = ∟ DFG. . . . . Const.
∴ ∟ ANB = ∟ DFE.
Similarly ∟ AMC = ∟ DEF.
∴ third ∟ L = third ∟ D . . . . Th. 8.
∴ △ LMN, circumscd. about ⊙ ABC, is equiang. to △ DEF. Q.E.F.

Exercises.

On the Construction of Triangles.

1121. Construct a △, having given its base, vert. ∠ and a side.

1122. Construct a △, having given its base, vert. ∠ and the pt. in the base on which the ⊥ falls from the vertex. (Bomb. Mat.)

1123. Construct a △, having given its base, vert. ∠ and altitude. (Bomb. Sch. Fin.)

1124. Construct a △, having given its base, vert. ∠ and area. (Allah. Mat.)

1125. Construct the △ whose area is a maximum, having given its base and vert. ∠. (Bomb. Mat.)

1126. Construct a △, having given its base, vert. ∠ and the length of the median bisecting the base.

* 1127. Construct a △, having given its base, vert. ∠ and the pt. where the base meets the internal bisector of the vert. ∠.

* 1128. Construct a △, having given its base, vert. ∠ and the median through either extremity of the base. (Mad. Mat.)

* 1129. Construct a △, having given its base, vert. ∠ and sum of remaining sides.

* 1130. Construct a △, having given its base, vert. ∠ and difference of remaining sides.

CIRCLES AND REGULAR POLYGONS.

Additional Problem I.

Gen. Enun. *In a given circle to inscribe a regular polygon of* n *sides.*

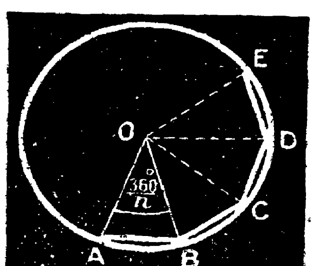


Fig. 290a.

Part. Enun. Let O be the centre of a given ⊙.

It is reqd. to inscribe a reg. polygon of n *sides in the given* ⊙.

Const. Draw two radii OA, OB containing an $\angle$ AOB = $\frac{360°}{n}$.

(Note. This can be done with ruler and compasses for only particular values of n. For other values a protractor may be used.)

Join AB and set off chords BC, CD, DE . . . each equal to AB round the circumference.

Then shall ABCDE . . . be a reg. polygon of n sides insd. in the given ⊙.

Proof. Join OC, OD, OE.

∵ chord AB = chord BC = chord CD = chord DE = . . . Const.

∴ minor arc AB = minor arc BC = minor arc CD = minor arc DE = ... Th. 38 (A).

∴ $\angle$AOB = $\angle$BOC = $\angle$COD = $\angle$DOE = . . . Th. 37 (B)

But $\angle$ AOB = $\frac{360°}{n}$ . . . . Const.

∴ each of these angles = $\frac{360°}{n}$, so that n of them make up 360°.

Hence the polygon has n sides.

And it is evidently equilat.

Again, ∵ $\angle$ AOC = $\angle$ BOD $\left(\text{each} = \frac{720°}{n}\right)$.

∴ reflex $\angle$ AOC = reflex $\angle$ BOD.

But reflex $\angle$ AOC = 2$\angle$ ABC and reflex $\angle$ BOD = 2$\angle$ BCD. Th. 42.

∴ $\angle$ABC = $\angle$ BCD.

Similarly, $\angle$ BCD = $\angle$ CDE = . . .

Hence the polygon is also equiang. and, therefore, regular. Q.E.F.

Cor. *Inscribe a regular figure of* 3, 4, 6, 8, 12, 24 *sides in a given* ⊙.

ADDITIONAL PROBLEM II.

Gen. Enun. *About a given circle to circumscribe a regular polygon of* n *sides.*

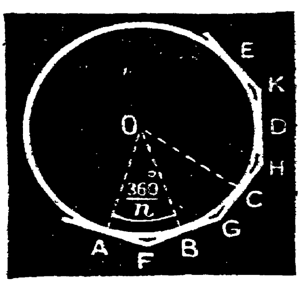


FIG. 290B.

Part. Enun. Let O be the centre of a given ⊙.

It is reqd. to circumscribe a reg. polygon of n *sides about the given* ⊙.

Const. Determine the angular pts. A, B, C, D, E . . . of a reg. polygon of n sides inscd. in the given ⊙ Addit. Prob. I.
At the pts. A, B, C, D, E . . . draw tangents to the ⊙ intersecting at F, G, H, K . . . . Prob. 16 (A).
Then shall FGHK . . . be a reg. polygon of n sides circumscd. about the given ⊙.

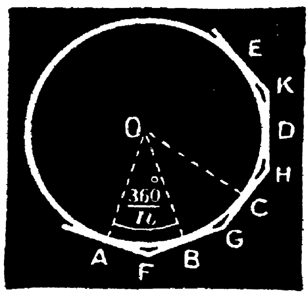


FIG. 290B.

Proof. Join OA, OB, OC.

$$\because \begin{cases} \angle \text{AOB} = \angle \text{BOC} \left(\text{each} = \dfrac{360°}{n}\right) \\ \text{OA} = \text{OB} = \text{OC (radii)} \\ \angle\text{s OAF, OBF, OBG, OCG are all rt. } \angle\text{s.} \quad \text{Th. 40} \end{cases}$$

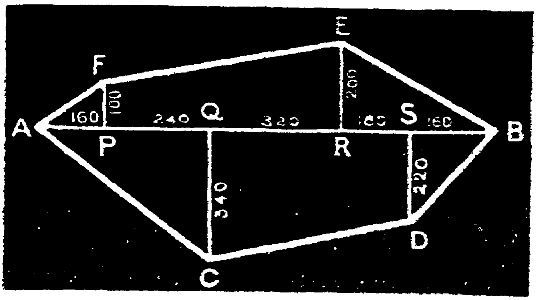
∴ fig OAFB can be made to coincide with fig. OBGC by superposition.
∴ AF = BG.
But AF = BF . . . . Th. 40, Cor 5.
Hence BF = BG = CG = CH = DH = DK = . . .
∴ FG = GH = HK = . . .
∴ the polygon is equilat.
And it is evidently of n sides.
Again, ∵ ∠ AFB coincides with ∠ BGC when fig. OAFB is made to coincide with fig. OBGC.
∴ ∠ AFB = ∠ BGC.
Similarly, ∠ BGC = ∠ CHD = ∠ DKE = . . .
Hence the polygon is also equiang. and, therefore, regular. Q.E.F.

Cor. *Circumscribe a regular figure of* 3, 4, 6, 8, 12, 24 *sides about a given circle.*

ON THE PRACTICAL DETERMINATION OF THE AREAS OF IRREGULAR RECTILINEAL FIGURES.

FIRST METHOD—THE FIELD BOOK.

Consider the irregular rectilineal figure ACDBEF and the accompanying notes taken from a field-book. The entries in the central column give measurements from A along the base line AB and must be read upwards. The entries in the side columns give measurements along the offsets PF, QC, RE, SD, and each appears



	Links.	
	⊙ B	
0	1060	0
	900	220 D
E 200	720	
	400	340 C
F 100	160	
0	0	0

From ⊙ A go east.

Fig. 291.

to the right or left according as the offset to which it refers springs to the right or left of the base line as a man walks from A to B. The figure 0 in a side column indicates that the corresponding point on the boundary is at no distance from the base line; in other words, that at this point the base line meets the boundary.

Thus, these notes may be amplified as follows:—

	Arrive at Station B.	
	AB = 1060 links	
	AS = 900 „	SD = 220 links
RE = 200 links	AR = 720 „	
	AQ = 400 „	QC = 340 „
PF = 100 „	AP = 160 „	
	Start from Station A and go east.	

Given the notes, it is easy to draw the figure, and its area is determined by finding the areas of the triangles and trapeziums into which it is divided by the base line and offsets thus:—

Fig. ACDBEF = $\triangle$ AQC + trap. QSDC + $\triangle$ SBD + $\triangle$ BRE + trap. ERPF + $\triangle$ FPA.

$\therefore$ reqd. area = $\{\frac{1}{2} \times 400 \times 340 + \frac{1}{2} \times 560 \times 500 + \frac{1}{2} \times 160 \times 220 + \frac{1}{2} \times 340 \times 200 + \frac{1}{2} \times 300 \times 560 + \frac{1}{2} \times 160 \times 100\}$ sq. links.

= 351600 sq. links = 35·16 sq. chains.

It is sometimes convenient to use two or more base lines. Consider, for example, the irregular rectilineal figure ADEBFCG and the accompanying notes taken from a field-book.

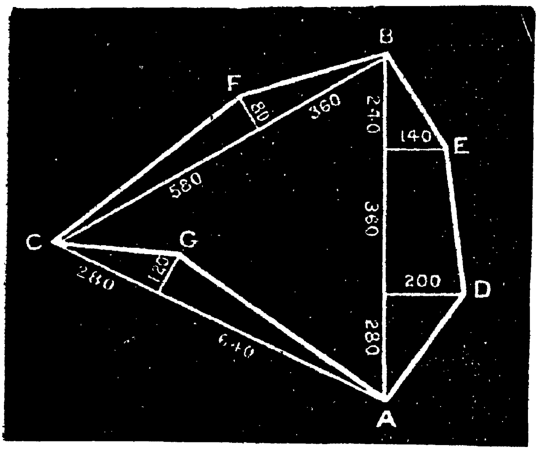


FIG. 292.

	Links.	
	To ⊙ A	
0	920	
G 120	280	
0	0	
	Turn to the left.	
	To ⊙ C	
	940	0
	360	80 F
	0	0
	Turn to the left.	
	To ⊙ B	
	880	0
	640	140 E
	280	200 D
	0	0

From ⊙ A go north.

In this and similar cases we first find the area of the figure bounded by the base lines and then add the areas that fall outside and subtract the areas that fall inside this figure thus :—

Fig. ADEBFCG = $\triangle$ ABC + fig. ADEB + $\triangle$ BFC − $\triangle$ AGC.

$\therefore$ reqd. area = $(\sqrt{1370 \times 490 \times 430 \times 450} + 106000 + 37600 - 55200)$ sq. links = 44·8811 sq. chains nearly.

Notice that "turn to the left" is not necessarily at right angles.

This method is used by land surveyors. They regard the area they wish to survey as bounded by a rectilineal figure which nearly coincides with the actual boundary. To ensure greater accuracy, two or more surveys of the same area are made, varying the base lines, and the mean result taken.

SECOND METHOD—TRIANGULATION.

This method amounts to dividing the figure into triangles by

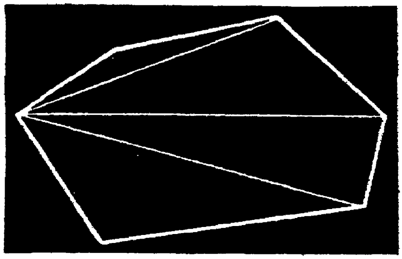


FIG. 293.

drawing diagonals and finding the areas of these triangles by making suitable measurements.

THIRD METHOD—REDUCTION TO A TRIANGLE.

By the construction of Prob. 13 a rectilineal figure of any number of sides can be reduced to an equivalent rectilineal figure of one less number of sides. By repeating this construction any irregular rectilineal figure can be reduced to a triangle of equal area, and this area can be determined by making suitable measurements.

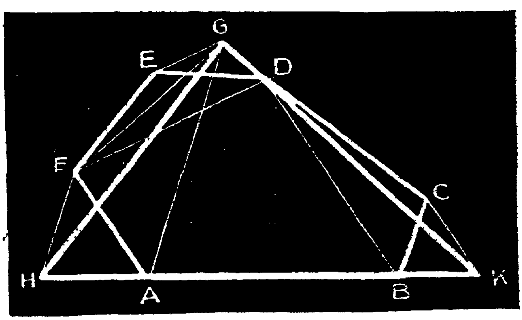


FIG. 294.

For example, figure ABCDEF = figure ABCGF = figure HBCG = figure HKG.

Exercises.

1131. Plan a field to scale 400 yds. = 1 in. from the accompanying notes, and find its area in acres, roods and poles.

	Yards.	
	To ⊙ D	
0	1200	0
C 200	700	
	500	100 B
0	0	0
	From ⊙ A go N.	

1132. Plan a field to scale R. F. = $\frac{1}{15840}$ from the accompanying notes, and find its area in acres, roods and poles.

	Yards.	
	To ⊙ D	
0	1500	0
C 250	1100	
	800	375 E
B 125	650	
0	0	0
	From ⊙ A go E.	

1133. Plan a field to scale 70 yds. = 1 in. from the accompanying notes, and find its area in sq. yds.

	Yards.	
	To ⊙ B	
0	210	112 F
D 64	140	
C 86	60	
0	0	48 E
	From ⊙ A go W.	

1134. Plan a field to scale 4 chains = 1 in. from the accompanying notes, and find its area in acres.

	Links.	
	To ⊙ G	
0	1200	0
F 425	950	
	730	40 E
D 350	500	
C 60	420	
	275	280 B
0	0	0
	From ⊙ A go N.E	

1135. Plan a field to scale R. F. = $\frac{1}{3960}$ from the accompanying notes, and find its area in acres, roods and poles.

	Links.	
	To ⊙ B	
0	1500	140
240	1050	
	980	60
	740	130
280	700	
	520	170
200	420	
	360	80
320	210	
290	0	170
	From ⊙ A go N.	

1136. Plan a field ACDEBFGH to scale 150 yds. = 1 in. from the accompanying notes, in which the offsets to a stream are given in italics; also find its area in sq. yds.

	Yards.	
	To ⊙ B	
0	420	0
E 94	380	
D 120 *52*	330	
Stream crosses	316	base line
	284	*88* F
	246	*120* 260 G
	162	*170* H
C 46	57	
0	0	0
	From ⊙ A go S.W.	

1137. Plan a field to scale R. F. = $\frac{1}{1584}$ from the accompanying notes, and find its area in acres.

	Links.	
	To ⊙ A	
0	500	
H 25	300	
G 20	160	
	0	0
	Turn to the right.	
	To ⊙ C	
	400	0
	160	20 F
	0	0
	From ⊙ B go N.E.	
	To ⊙ B	
0	300	
E 10	180	
D 12	90	
0	0	
	From ⊙ A go N.W.	

1138. Plan a field to scale 2 chains = 1 in. from the accompanying notes, and find its area in acres.

	Links.	
	To ⊙ D	
	750	0
	630	160 C
E 120	300	
0	0	
	Turn to the right.	
	To ⊙ B	
	800	
	200	250 C
	0	0
	From ⊙ A go W.	

1139. Plan a field to scale R. F. = $\frac{1}{3168}$ from the accompanying notes, and find its area in acres.

	Links.	
	To ⊙ A	
0	850	
50	140	
0	70	30
	35	0
	0	0
	Turn to the left.	
	To ⊙ C	
	340	0
	75	28
	0	0
	Turn to the left.	
	To ⊙ B	
	1000	0
	340	50
	120	20
	0	0

From ⊙ A go E.

1140. The accompanying rectilineal figure is the plan of a field drawn to scale 100 yds. = 1 in.; copy the figure on tracing paper, draw a convenient base line and offsets and note down measurements as for a

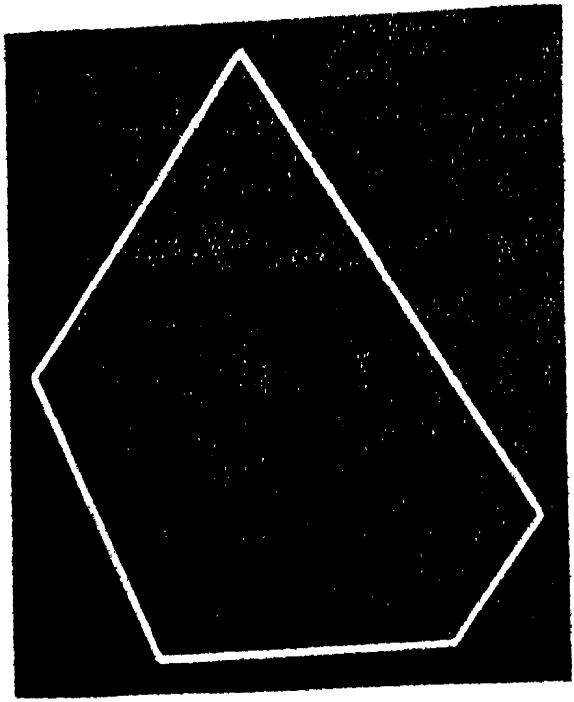


Fig. 295.

field book. Then find the area of the field in acres. Verify (1) by triangulation, (2) by reduction to a triangle.

PATTERN DRAWING.

The accompanying geometrical patterns are to be copied as exercises in neatness and accuracy. They are composed of circles, arcs of circles and straight lines, and can all be drawn with ruler and compasses. The form of each pattern suggests its construction lines, and these should be drawn in black lead, only those lines being inked in that compose the pattern. Each copy should be made from three to four inches in greatest width.

It is useful to bear in mind that when two arcs join so as to form one continuous curve, they are parts of circles touching one another at the point where the two arcs join, and so this point lies on

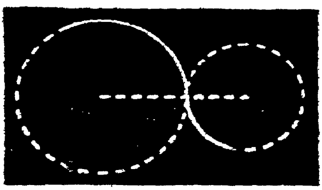


Fig. 296.

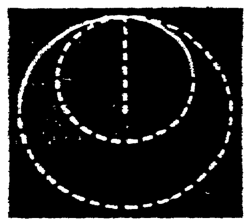


Fig. 297.

their line of centres (Th. 41). Also when a straight line joins an arc to form one continuous line, the arc is part of a circle touching

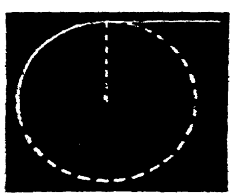


Fig. 298.

the straight line at the point where the straight line joins the arc, and so the perpendicular to the straight line at this point passes through the centre of the arc (Th. 40).

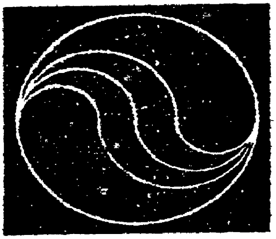


FIG. 299.

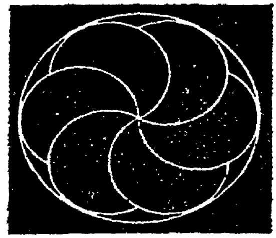


FIG. 300.

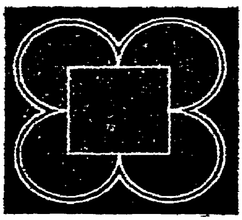


FIG. 301.

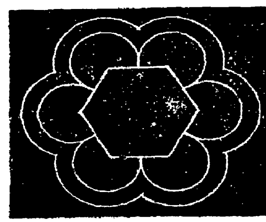


FIG. 302.

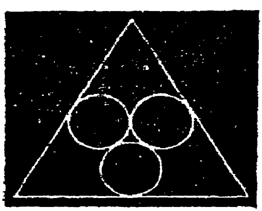


FIG. 303.

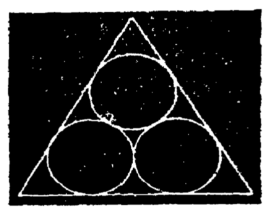


FIG. 304.

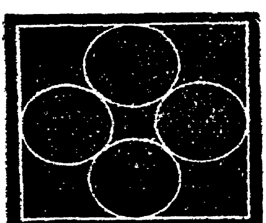


FIG. 305.

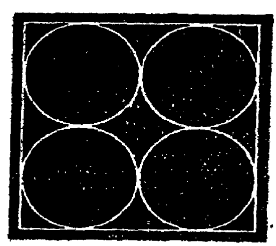


FIG. 306.

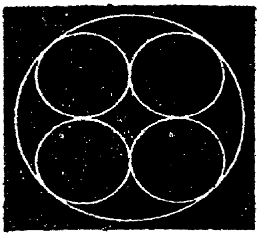


FIG. 307.

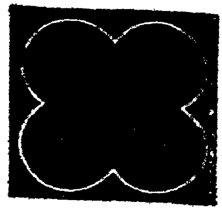


FIG. 308.

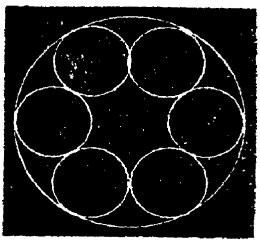


FIG. 309.

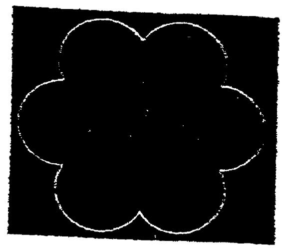


FIG. 310.

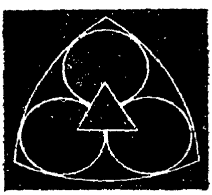


FIG. 311.

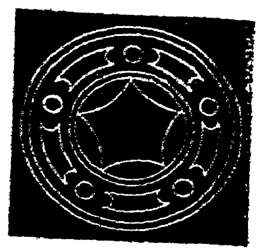


FIG. 312.

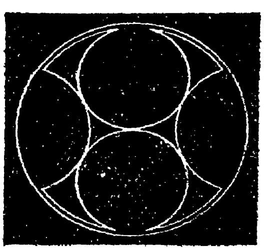


FIG. 313.

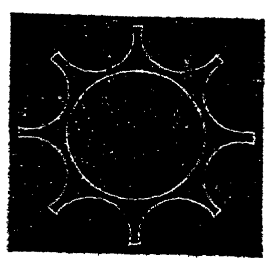


FIG. 314.

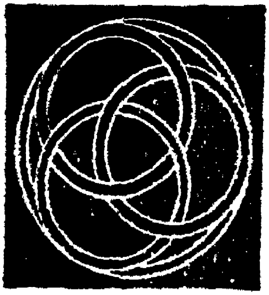


FIG. 315.

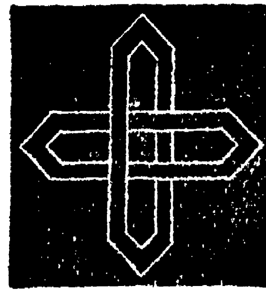


FIG. 316.

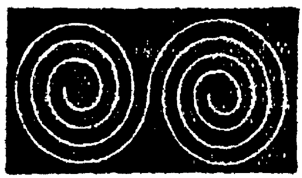


FIG. 317.

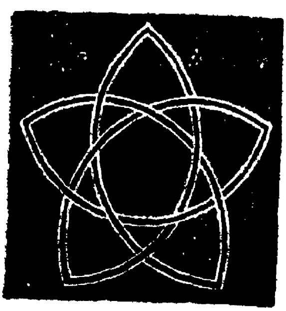


FIG. 318.

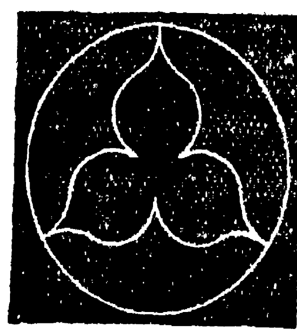


FIG. 319.

PART III.

EXPERIMENTAL SECTION.

PART III.

EXPERIMENTAL SECTION.

EXTENSION OF THE THEOREM OF PYTHAGORAS.

Exp. 351. ABC in Fig 320 is an *obtuse-angled* triangle drawn on squared paper and ACDE, ABGF, BCKH are squares described on its side AC opposite the *obtuse angle* and its sides AB, BC respectively. Copy Fig. 320 on squared paper, and show by counting

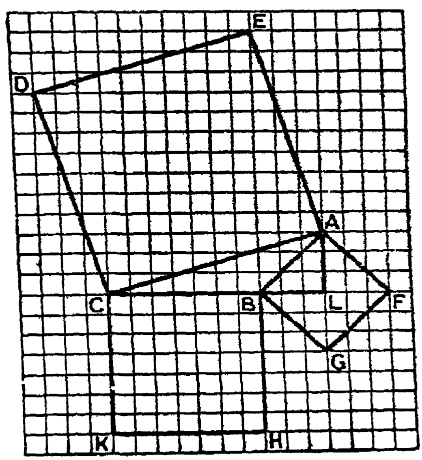


FIG. 320.

divisions that the square on AC is *greater than* the sum of the squares on AB, BC by twice the rectangle contained by BC and the projection on BC of AB (or by twice the rectangle contained by AB and the projection on AB of BC).

Working :—

$$AC^2 = AL^2 + CL^2 = 3^2 + 10^2 = 109$$
$$AB^2 = AL^2 + BL^2 = 3^2 + 3^2 = 18.$$
$$BC^2 = 7^2 = 49$$
$$\text{Twice rect. } BC.BL = 2 \times 7 \times 3 = 42$$
$$\text{And } 109 = 18 + 49 + 42$$

Exp. 352. Repeat Exp. 351 two or three times with obtuse-angled triangles of different measurements.

Exp. 353. ABC in Fig. 321 is *any* triangle drawn on squared paper and ACDE, ABGF, BCKH are squares described on its side AC opposite an *acute angle* and its sides AB, BC respectively. Copy Fig. 321 on squared paper and show by counting divisions

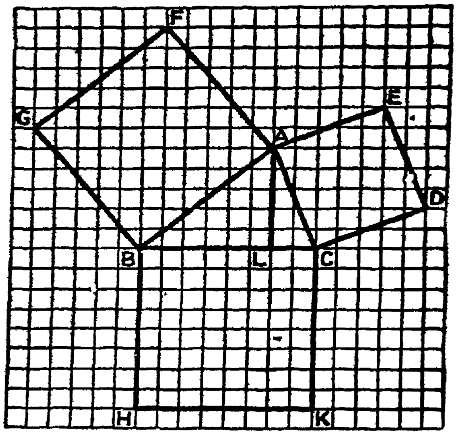


FIG. 321.

that the square on AC is *less than* the sum of the squares on AB, BC by twice the rectangle contained by BC and the projection on BC of AB (or by twice the rectangle contained by AB and the projection on AB of BC).

Working :—

$$AC^2 = AL^2 + LC^2 = 5^2 + 2^2 = 29$$
$$AB^2 = AL^2 + BL^2 = 5^2 + 6^2 = 61$$
$$BC^2 = 8^2 = 64$$
$$\text{Twice rect. } BC \cdot BL = 2 \times 8 \times 6 = 96$$
$$\text{And } 29 = 61 + 64 - 96$$

Exp. 354. Repeat Exp. 353 two or three times with any triangles of different measurements.

From Exps. 351-354 we are led to conclude :—

The square on a side of a triangle is greater or less than the sum of the squares on the other two sides, according as the angle contained by those sides is obtuse or acute, by twice the rectangle contained by either of the two sides and the projection on it of the other.

Learn this by heart.

Exp. 355. Make a triangle from the following data: AB = 5 cm., BC = 3·1 cm., CA = 7·5 cm., and calculate the length of the projection of BC on AB. Verify by drawing and measurement.

Exp. 356. Make a triangle from the following data: AB = 2·1 in., BC = 1·8 in., CA = 1·6 in., and calculate the length of the projection of AB on BC. Verify by drawing and measurement.

RECTANGLES UNDER SEGMENTS OF CHORDS.

Exp. 357. Describe a circle and through any point P inside the circle draw a chord AB. The point P is said to divide the chord AB *internally* into the segments PA, PB. Measure PA and PB and calculate, to the nearest tenth of a square inch, the area of the rectangle contained by PA and PB. Similarly draw other chords

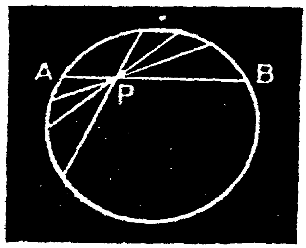


FIG. 322.

through P and calculate to the nearest tenth of a square inch the areas of the rectangles contained by the segments into which each chord is divided internally at P. Tabulate and compare your results.

Exp. 358. Describe a circle, and from any point P outside the circle draw a straight line PAB to cut the circle in A, B. The point P is said to divide the chord AB *externally* into the seg-

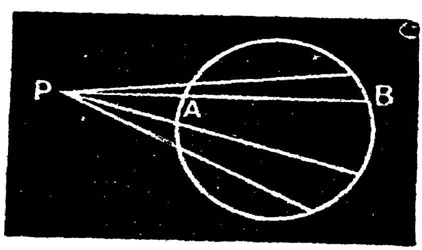


FIG. 323.

ments PA, PB. Measure PA and PB, and calculate, to the nearest tenth of a square inch, the area of the rectangle contained by PA

and PB. Similarly draw other straight lines from P to cut the circle, and calculate to the nearest tenth of a square inch the areas of the rectangles contained by the segments into which each chord is divided externally at P. Tabulate and compare your results.

From Exps. 357, 358, allowing for errors of measurement, we are led to conclude :—

If two chords of a circle intersect either inside or outside the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

Learn this by heart.

Exp. 359. From a point C in a straight line AB a straight line CD is drawn. AC = 2·4 cm., BC = 4·3 cm., CD = 3·5 cm. A circle is described through A, B, D and DC is produced to cut this circle at E. Calculate the length of CE and verify by drawing and measurement.

Exp. 360. From a point A two straight lines AB, AC are drawn and in AB a point D is taken. AB = 2·8 in., AC = 3·1 in., AD = 1·7 in. A circle is described through B, C, D and this circle cuts AC at E. Calculate the length of AE and verify by drawing and measurement.

Exp. 361. Illustrate by drawing figures, measurement and cal-

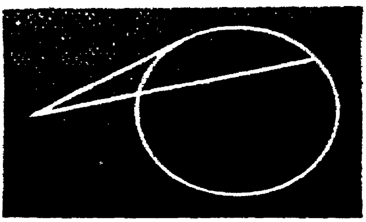


FIG. 324.

culation to the nearest tenth of a square inch the following truth :—

If from any point without a circle a chord and a tangent are drawn the rectangle contained by the segments of the chord is equal to the square on the tangent.

Learn this by heart.

Notice that this is the particular case of Exp. 358 when PA = PB (Fig. 323).

Exp. 362. Describe a circle and place in it a chord PQ = 1·1 in. Produce PQ to R so that QR = 1·3 in. Calculate to the nearest tenth of an inch the length of a tangent from R to the circle. Verify by drawing and measurement.

Exp. 363. Illustrate by drawing figures, measurement and cal-

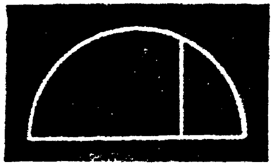


FIG. 325.

culation to the nearest tenth of a square inch the following truth :—

If from any point on the circumference of a semi-circle a perpendicular is drawn to the diameter, the square on the perpendicular is equal to the rectangle contained by the segments of the diameter.

Learn this by heart.

Notice that this is the particular case of Exp. 357 when one of the chords through **P** (Fig. 322) is a diameter of the circle and another is at right angles to it.

THE GRAPHICAL EXTRACTION OF ARITHMETICAL SQUARE ROOT.

Exp. 364. Draw a straight line AB = 3 in. and divide it internally at C so that AC = 2 in. and, therefore, BC = 1 in. On AB describe a semi-circle and at C draw CD perpendicular to AB meeting the semi-circle at D. Then CD = $\sqrt{2}$ in. Why? Calculate to the nearest hundredth of an inch the length of CD and verify by measurement.

Exp. 365. Draw a straight line AB = 6 cm. and divide it internally at C so that AC = 4 cm. and, therefore, BC = 2 cm. At C draw CD perpendicular to AB such that the square on CD = the rectangle contained by AC and BC. Verify by measurement and calculation.

Exp. 366. Draw the side of a square equal to a rectangle measuring 3 in. by 1 in. (Construction as in Exp. 364). Verify by measurement and calculation.

Exp. 367. Draw the side of a square equal to a rectangle measuring (1) 10 sq. cm., (2) 4·6 sq. in., (3) 6·2 sq. cm., (4) 680 sq. mm. (Construction as in Exp. 364). Verify by measurement and calculation.

Exp. 368. Represent graphically the following lengths: (1) $\sqrt{6}$ in., (2) $\sqrt{7}$ cm., (3) $\sqrt{21}$ cm., (4) $\sqrt{1200}$ mm. (Construction as in Exp. 364). Verify by measurement and calculation.

THE GRAPHICAL SOLUTION OF CERTAIN QUADRATIC EQUATIONS.

Exp. 369. Draw a straight line AB = 10 cm. and divide it internally at C so that the rectangle contained by AC and BC is equal to 16 sq. cm. This amounts to solving the simultaneous equations

$$\left.\begin{aligned} x + y &= 10 \\ xy &= 16 \end{aligned}\right\}$$

where x cm. = AC and y cm. = BC.

Or, what is equivalent, the quadratic equation,

$$x^2 - 10x + 16 = 0,$$

where x cm. = AC.

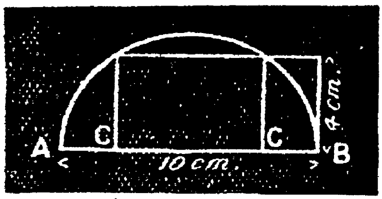


FIG. 326.

Fig. 326, on a reduced scale, indicates the construction The two positions of C correspond to the two roots of the quadratic.

Exp. 370. Taking ·1 in. as the unit of length, give a graphical solution of the simultaneous equations

$$\left.\begin{aligned} x + y &= 30 \\ xy &= 144 \end{aligned}\right\}.$$

Verify by solving algebraically.

Exp. 371. Taking 1 mm. as the unit of length, give a graphical solution of the quadratic equation

$$x^2 - 40x + 169 = 0.$$

Verify by solving algebraically.

Exp. 372. Taking 1 in. as the unit of length, give a graphical solution of the quadratic equation

$$x^2 - 4x + 2 = 0.$$

Verify by solving algebraically.

Note. 2 is not a perfect square, but $\sqrt{2}$ in. can be found either arithmetically or as in Exp. 364.

Exp. 373. Taking 1 cm. as the unit of length, give graphical solutions of the quadratic equations

$$\text{(i) } x^2 - 6x + 5 = 0.$$
$$\text{(ii) } x^2 - 6x + 9 = 0.$$

Verify by solving algebraically.

THEORETICAL SECTION.

RECTANGLES UNDER SEGMENTS OF STRAIGHT LINES.

THEOREM A.

(Euc. II. 1.)

Gen. Enun. *If there are two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line.*

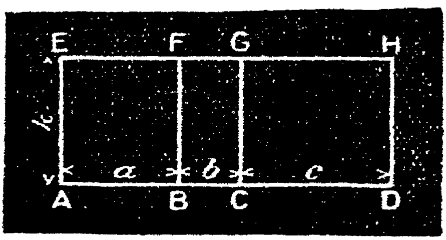


FIG. 327.

Illustration. Let AD be a st. line divided into 3 parts AB, BC, CD measuring a, b, c of the same linear unit respy., and let AE be another st. line drawn $\perp$ to AD and measuring k of the same linear unit.

Complete the rects. ADHE, ABFE, BCGF, CDHG.

Now

rect. ADHE = rect. ABFE + rect. BCGF + rect. CDHG.

But

rect. ADHE = $k(a + b + c)$ of the corresponding sq. unit. Th. 26.

" ABFE = ka " " " "

" BCGF = kb " " " "

" CDHG = kc " " " "

$\therefore k(a + b + c) = ka + kb + kc.$

Or AE(AB + BC + CD) = AE.AB + AE.BC + AE.CD.

Cor. 1. *If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the rectangles contained by the whole line and each of the parts* (Euc. II. 2).

Corresponding algebraical identity

$$(a+b)^2 = a(a+b) + b(a+b).$$

Cor. 2. *If a straight line is divided into any two parts, the rectangle contained by the whole line and one part is equal to the square on that part together with the rectangle contained by the two parts* (Euc. II. 3).

Corresponding algebraical identity

$$a(a+b) = a^2 + ab.$$

Exercises.

1141. If in Fig. 327 AD = 4·3 cm., AE = 2·5 cm., ACGE = 8 sq. cm., find CD.

1142. Draw diagrams on squared paper to illustrate the following algebraical identities :—

(i) $x(x+6) = x^2 + 6x$.
(ii) $x(y+5) = xy + 5x$.
(iii) $(x+3)(x+7) = x^2 + 10x + 21$.
(iv) $x(x-3) = x^2 - 3x$. $(x > 3.)$
(v) $(x-3)(x+2) = x^2 - x - 6$. $(x > 3.)$

1143. Draw diagrams to illustrate the following algebraical identities and state the corresponding geometrical theorems :—

(i) $ab = ba$.
(ii) $(a-b)c = ac - bc$. $(a > b.)$
(iii) $(a+b)(c+d) = ac + ad + bc + bd$.
(iv) $(a-b)(c+d) = ac + ad - bc - bd$. $(a > b.)$
(v) $(a-b)(c-d) = ac - ad - bc + bd$. $(a > b, c > d)$.

1144. The hypot. BC of a rt ∟d △ ABC is divided at D so that BC . CD = AC^2; show that BC . BD = AB^2. (Bomb. Prev.)

* 1145. *AB is divided at* H *so that* AB . BH = AH^2 *and* L *is taken in* AH *so that* HL = HB ; *prove that* AH . AL = HL^2. (Bomb. Sch. Fin.)

* 1146. *If* A, B, C, D *are 4 pts. taken in order, in a st. line, prove that* AC . BD = AB . CD + AD . BC. (Bomb. Sch. Fin.)

* 1147. If P is the orthocentre of an acute-angled △ ABC, prove that AP . BC + BP . CA + CP . AB = 4 △ ABC. (Cal. Mat.)

* 1148. AB is a st. line trisected in C, D and produced to E ; show that CE . ED = AE . EB + $2AC^2$. (Bomb. Prev.)

* 1149. ABC is an isos. △ rt. ∟d at A ; D is a pt. in BC and E in BC produced such that AD is ⊥ to AE ; prove that BD . CE + BE . CD = 2AD . AE. (Mad. F. E.)

Theorem B.

(Euc. II. 4.)

Gen. Enun. *If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the two parts.*

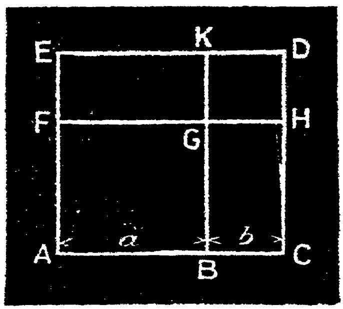


Fig. 328.

Illustration. Let AC be a st. line divided into 2 parts AB, BC measuring a, b of the same linear unit respy.

On AC construct the sq. ACDE.

In AE take a pt. F such that AF = AB.

Draw BGK, FGH ∥ to the sides of the sq. ACDE.

Now

sq. ACDE = sq. ABGF + sq. GHDK + rect. BCHG + rect. FGKE.

But

sq. ACDE = $(a+b)^2$	of the	corresponding	sq. unit.	Th. 26, Cor. 1.
„ ABGF = a^2	„	„	„	„
„ GHDK = b^2	„	„	„	„
rect. BCHG = ab	„	„	„	Th. 26.
„ FGKE = ab	„	„	„	

$\therefore (a+b)^2 = a^2 + b^2 + 2ab.$

Or $(AB + BC)^2 = AB^2 + BC^2 + 2AB.BC.$

Exercises.

1150. If in Fig. 328 ACDE = 6·25 sq. in., ABGF = 3·24 sq. in., find FGKE.

1151. Draw a diagram on squared paper to illustrate the algebraical identity

$$(x + 9)^2 = x^2 + 18x + 81.$$

1152. Draw diagrams to illustrate the following algebraical identities, and state the corresponding geometrical theorems :—

(i) $(3a)^2 = 9a^2$.
(ii) $(a + 2b)^2 = a^2 + 4ab + 4b^2$.
(iii) $(2x + 3y)^2 = 4x^2 + 9y^2 + 12xy$.
(iv) $(p + q + r)^2 = p^2 + q^2 + r^2 + 2pq + 2pr + 2qr$.

1153. *The sq. on any st. line is 4 times the sq. on half the line.*

1154. If AB is divided at C so that $AB^2 = AC^2 + 2BC^2$, show that BC = 2AC.

1155. *In a* △ ABC, AD *is drawn* ⊥ *to* BC; *if* BD . DC = AD^2, *prove that* ∟ BAC *is a rt.* ∟. (Bomb. Mat.)

1156. AO is ⊥ to the base BOC of a △ ABC; prove that $AB^2 + AC^2 + 2BO . OC = BC^2 + 2AO^2$. (Mad. Mat.)

1157. *In a rt.* ∟*d* △ *if a* ⊥ *is drawn from the rt.* ∟ *to the hypot., the sq. on this* ⊥ = *the rect. contained by the segments of the hypot.* (Bomb. Mat.)

1158. *From the rt.* ∟ A *of a rt.* ∟*d* △ ABC, AD *is drawn* ⊥ *to* BC; *prove that* AB^2 = BC . BD. (Mad. Mat.)

1159. ABC is a △ rt. ∟d at C; prove that AB^2 + 4 △ ABC = the sq. on the line made up of AC and CB. (Bomb. Mat.)

1160. ABC is an isos. △, CA, CB being the = sides; BO is ⊥ to BC meeting CA produced in O; show that $OB^2 = OA^2 + 2OA . AC$. (Cal. F. E.)

1161. ABC is an isos. △, CA, CB being the = sides: BO is ⊥ to CA; show that $BO^2 = 2CO . OA + OA^2$.

1162. A rect. ABCD has the side AB produced to E so that BE = BC; show that rect. ABCD = ½ difference of sqs. on AE and BD. (Allah. Mat.)

* 1163. AC is divided at B and produced to D; prove that $AC^2 + BD^2 = AD^2 + BC^2 - 2AB . CD$.

* 1164. PS is divided at R and produced to Q; prove that $PR^2 + SQ^2 = PS^2 + RQ^2 - 2PQ . RS$.

* 1165. If AB is divided at C so that AB . BC = AC^2 and if AB is produced both ways to D and E so that AB = BD and CA = AE, show that BE^2 = AB . DE. (Bomb. Prev.)

* 1166. AB is divided at H so that AB . BH = AH^2; AB is produced to F so that BF = 2AH; prove that $AF^2 = 5AB^2$. (Mad. F. A.)

Theorem C.

(Euc. II. 7.)

Gen. Enun. *If a straight line is divided into any two parts, the square on one of the parts is equal to the sum of the squares on the whole line and on the other part diminished by twice the rectangle contained by the whole line and that part.*

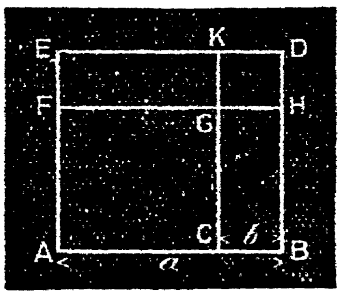


Fig. 329.

Illustration. Let AB be a st. line divided into two parts AC, CB measuring $(a - b)$, b of the same linear unit respy.

On AB construct the sq. ABDE.

In AE take a pt. F such that AF = AC.

Draw CGK, FGH ∥ to the sides of the sq. ABDE.

Now

sq. ACGF = sq. ABDE + sq. GHDK − rect. CBDK − rect. FHDE.

But

sq. ACGF	$= (a - b)^2$	of the	corresponding	sq. unit.	Th. 26, Cor. 1.
,, ABDE	$= a^2$	,,	,,	,,	,,
,, GHDK	$= b^2$	,,	,,	,,	,,
ect. CBDK	$= ab$	,,	,,	,,	Th. 26.
,, FHDE	$= ab$	,,	,,	,,	,,

$\therefore (a - b)^2 = a^2 + b^2 - 2ab.$

Or $(AB - BC)^2 = AB^2 + BC^2 - 2AB . BC.$

Def. 66. If a point B is taken in a straight line AC, then AC is said to be divided internally at B into the segments AB and BC.

FIG. 330.

If a point B is taken in a straight line AC *produced*, then AC is

FIG. 331.

said to be divided externally at B into the segments AB and BC.

Hence the enunciations of Theorems B and C may be combined thus:—

If a straight line is divided *internally* / *externally* *at any point, the square on the given line is equal to the sum of the squares on the two segments* *together with* / *diminished by* *twice the rectangle contained by the segments.*

Exercises.

1167. If in Fig. 329 AB = 3·3 cm., ACGF = 4·41 sq. cm., find CBDK.

1168. Draw a diagram on squared paper to illustrate the algebraical identity

$$(x-5)^2 = x^2 - 10x + 25. \quad (x > 5.)$$

1169. Draw diagrams to illustrate the following algebraical identities and state the corresponding geometrical theorems:—

(i) $(a-2b)^2 = a^2 + 4b^2 - 4ab. \quad \left(b < \frac{a}{2}.\right)$

(ii) $(2x-y)^2 = 4x^2 + y^2 - 4xy. \quad \left(x > \frac{y}{2}.\right)$

(iii) $(a+b-c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc \quad (\overline{a+b} > c.)$

1170. *The rect. contained by 2 unequal st. lines $<\frac{1}{2}$ the sum of the sqs. on them.*

1171. *If* AB *is divided at* H *so that* $AB . BH = AH^2$, *then* $AB^2 + BH^2 = 3AH^2$. (Bomb. Mat.)

1172. If AB is divided at C so that $AB . BC = AC^2$, prove that the sq. on the line made up of AB and $BC = 5AC^2$. (Bomb. Mat.)

Theorem D.

Gen. Enun. *The difference of two squares is equal to the rectangle contained by the sum and difference of their sides.*

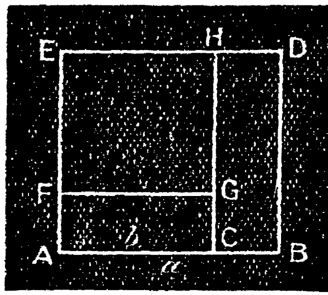


Fig. 332.

Illustration. Let AB be a st. line divided internally into 2 parts AC, CB measuring b, $(a - b)$ of the same linear unit respy.

On AB construct the sq. ABDE.

In AE take a pt. F such that EF = AC.

Draw CGH, FG ‖ to the sides of the sq. ABDE.

Now

sq. ABDE − sq. FGHE = rect. ACGF + rect. CBDH.

But

sq. ABDE $= a^2$ of the corresponding sq. unit. Th. 26, Cor. 1.

„ FGHE $= b^2$ „ „ „ „

rect. ACGF $= b(a-b)$ „ „ „ Th. 26.

„ CBDH $= a(a-b)$ „ „ „ „

$\therefore a^2 - b^2 = b(a-b) + a(a-b)$.

$= (a+b)(a-b)$ . . . . . Th. A.

Or $AB^2 - AC^2 = (AB + AC)(AB - AC)$.

Cor. 1. *If a straight line is divided into two equal parts and also into two unequal parts, the rectangle contained by the unequal parts and the square on the line between the points of section are together equal to the square on half the line.* (Euc. II. 5.)

Cor. 2. *If a straight line is bisected and produced to any point, the rectangle contained by the whole line thus produced and the part produced, together with the square on half the bisected line, is equal to the square on the line made up of the half and the part produced.* (Euc. II. 6.)

Exercises.

1173. If in Fig. 332 ABDE = 9·61 sq. in., CBDH = 2·17 sq. in., find ACGF.

1174. Draw a diagram on squared paper to illustrate the algebraical identity

$$x^2 - 49 = (x + 7)(x - 7). \quad (x > 7.)$$

1175. Draw a diagram to illustrate the following algebraical identity and state the corresponding geometrical theorem :—

$$a^2 - 4b^2 = (a + 2b)(a - 2b). \quad \left(b < \frac{a}{2}\right)$$

1176. *In a rt. ∟d △ the rect. contained by the sum and difference of the hypot. and one side = the sq. on the other side.*

1177. In any △ if the sq. on one side = the rect. contained by the sum and difference of the other two, the △ is rt. ∟d.

1178. AB is bisected at C and produced to D; prove that $AD^2 - DB^2 = 4AC \cdot CD$.

1179. AB is divided at H so that $AB \cdot BH = AH^2$ and K is the mid. pt. of AH; prove that AH, KH, KB are the sides of a rt. ∟d △. (Cal. F. E.)

1180. AD is divided equally at B and unequally at C so that $AC \cdot CD = 3BC^2$; prove that BD = 2BC.

* 1181. The base QR of an isos. △ PQR is divided internally or externally at S. Join PS, and prove that $QS \cdot SR$ = the difference between PS^2 and PQ^2.

* 1182. ABCD is a st. line. If AB = BC and AC = CD, then $AD \cdot DC = 8AB^2$.

* 1183. R is the mid. pt. of PQ; PQ is produced to S and QP to T; prove that the difference between $PS \cdot QS$ and $PT \cdot QT$ = the difference between RS^2 and RT^2.

* 1184. FEABCD is a st. line such that A is the mid. pt. of FC as well as of EB; prove that $BC \cdot CE + CD \cdot DF = BD \cdot DE$. (Bomb. Sch. Fin.)

* 1185. *If a rect. and a sq. have the same perim., the sq. will have the larger area.* (Mad. Mat.)

* 1186. If from the mid. pt. of one of the sides of a rt. ∟d △ a ⊥ is drawn to the hypot., the difference of the sqs. on the segments into which it is divided = the sq. on the other side.

* 1187. APQRS is a st. line such that AP = PS; prove that $AQ \cdot QS > AR \cdot RS$.

* 1188. *The difference of the sqs. on 2 sides of a △ = twice the rect. contained by the third side and the projection on the third side of the median that bisects the third side.*

* 1189. *The locus of the vertices of all the △s which have the same base and the difference of the sqs. on their sides = a given sq., is a st. line ⊥ to the base.*

* 1190. ABCD is an isos. trapezium having AB ∥ to CD; prove that $AB \cdot CD = AC^2 - AD^2$. (Bomb. Mat.)

* 1191. A, B, C, D, E are pts. on a st. line such that AB = BC = CD = DE and O is any pt. outside the st. line; prove that the difference of the sqs. on OA and OE is twice the difference of the sqs. on OB and OD. (Bomb. Mat.)

* 1192. BC *is drawn* $\perp$ *to* AB *and* = $\frac{1}{2}$AB; *from* CA, CD *is cut off* = CB *and from* AB, AE *is cut off* = AD; *prove that* AB . BE = AE^2. (Mad. F. A.)

Def. 67. If a st. line AB is divided internally or externally at E

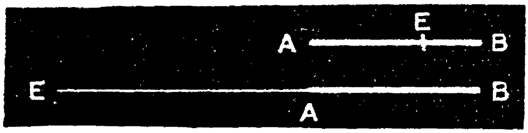


Fig. 333.

so that AB . BE = AE^2 then the st. line AB is said to be divided at E in medial section.

THEOREM K.

Gen. Enun. *The square on the sum of two straight lines exceeds the square on their difference by four times the rectangle contained by the lines.*

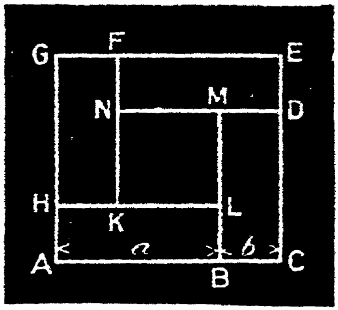


FIG. 334.

Illustration. Let AC be a st. line divided internally into 2 parts AB, BC measuring a, b of the same linear unit respy. Suppose $a > b$.

On AC construct the sq. ACEG.

Take pts. D in CE, F in EG, H in GA, such that CD = EF = GH = AB.

Draw BLM, DMN, FNK, HKL || to the sides of the sq. ACEG.

Now

sq. ACEG − sq. KLMN = rect. ABLH + rect. BCDM + rect. NDEF + rect. HKFG.

But

sq. ACEG = $(a + b)^2$ of the corresponding sq. unit. Th. 26, Cor. 1.

„ KLMN = $(a - b)^2$ „ „ „ „

rect. ABLH = rect. BCDM = rect. NDEF = rect. HKFG = ab of the corresponding sq. unit Th. 26.

$\therefore (a + b)^2 - (a - b)^2 = 4ab.$

Or $(AB + BC)^2 - (AB - BC)^2 = 4AB.BC.$

Cor. *If a straight line is divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the line made up of the whole and the first part.* (Euc. II. 8.)

Exercises.

1193. If in Fig. 334 ABLH = 1·14 sq. cm., BC = ·6 cm., find KLMN.

1194. Draw a diagram on squared paper to illustrate the algebraical identity

$$(x+3)^2 - (x-3)^2 = 12x. \quad (x > 3.)$$

Theorem F.

Gen. Enun. *The square on the sum of two straight lines and the square on their difference are together equal to twice the sum of the squares on the given lines.*

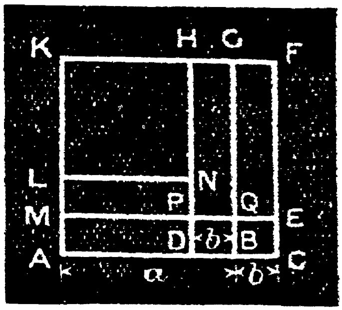


Fig. 335.

Illustration. Let AC be a st. line divided internally into 2 parts AB, BC measuring a, b of the same linear unit respy. Suppose $a > b$.

On AC construct the sq. ACFK.

Take pts. D in AB, M in AK, L in MK such that BD = AM = ML = BC.

Draw BQG, DPNH, MPQE, LN || to the sides of the sq. ACFK.

Now

sq. ACFK + sq. LNHK = 2 sq. MQGK + 2 sq. BCEQ.

But

sq. ACFK = $(a + b)^2$ of the corresponding sq. unit. Th. 26, Cor. 1.
" LNHK = $(a - b)^2$ " " " "
" MQGK = a^2 " " " "
" BCEQ = b^2 " " " "

$\therefore (a + b)^2 + (a - b)^2 = 2a^2 + 2b^2$.

Or $(AB + BC)^2 + (AB - BC)^2 = 2AB^2 + 2BC^2$.

Cor. 1. *If a straight line is divided equally and also unequally, the sum of the squares on the two unequal parts is twice the sum of the squares on half the line and on the line between the points of section.* (Euc. II. 9.)

Cor. 2. *If a straight line is bisected and produced to any point, the sum of the squares on the whole line thus produced and on the part produced is twice the sum of the squares on half the line bisected and on the line made up of the half and the part produced.* (Euc. II. 10.)

Exercises.

1195. If in Fig. 335 ACFK $= 5{\cdot}76$ sq. in., LNHK $= 1{\cdot}96$ sq. in., find QEFG.

1196. Draw a diagram on squared paper to illustrate the algebraical identity

$$(x+1)^2 + (x-1)^2 = 2x^2 + 2. \quad (x > 1.)$$

* 1197. *AB is divided equally at P and unequally at Q; prove that* $AQ^2 + QB^2 = 2AQ \cdot QB + 4PQ^2$.

* 1198. The vert. $\angle$ A of an isos. $\triangle$ ABC is a rt. $\angle$; the base BC is produced to any pt. D; prove that $2AD^2 = BD^2 + CD^2$. (Mad. Mat.)

* 1199. EF is divided at G and H, K are the mid. pts. of EG, GF; prove that $EK^2 + 3FK^2 = FH^2 + 3EH^2$. (Mad. Mat.)

* 1200. If the vertex A of an equilat. $\triangle$ is joined to a pt. D in the base BC, prove that $AD^2 = BD^2 + CD^2 + BD \cdot CD$. (Mad. F. A.)

EXTENSION OF THE THEOREM OF PYTHAGORAS.

THEOREM 49.

(Euc. II. 12.)

Gen. Enun. *In an obtuse-angled triangle the square on the side opposite the* obtuse *angle is* greater than *the sum of the squares on the sides containing the obtuse angle by twice the rectangle contained by either of these sides and the projection on it of the other.*

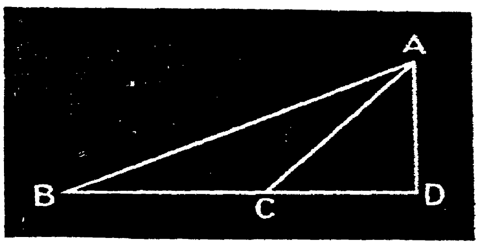


FIG. 336.

Part. Enun. Let ABC be an obtuse-angled $\triangle$ having $\angle$ C an obtuse $\angle$, and suppose AD drawn $\perp$ to BC produced so that CD is the projection on BC of AC.

It is reqd. to prove that

$$AB^2 = BC^2 + CA^2 + 2 . BC . CD.$$

Proof. $\because \angle CDA$ is a rt. $\angle$.

$\therefore AB^2 = BD^2 + DA^2$ . . . . . Th. 32.

$= (BC + CD)^2 + DA^2$.

$= BC^2 + CD^2 + 2BC . CD + DA^2$. Th. B.

But $CD^2 + DA^2 = CA^2$ . . . . . Th. 32.

$\therefore AB^2 = BC^2 + CA^2 + 2BC . CD$. Q.E.D.

Exercises.

1201. The side BC of an equilat. $\triangle$ ABC is produced to D and AD is joined; show that $AD^2 = BC^2 + CD^2 + BC . CD$. (Bomb. Prev.)

* 1202. ABC is an isos. $\triangle$ having $\angle$ A a rt. $\angle$ and D a pt. in CB produced; if $AD^2 = DB^2 + 3BA^2$, prove that DB = BC. (Mad. Mat.)

* 1203. The sides of a $\triangle$ are 3, 5, 7 ft. respy.; find the greatest $\angle$ of the $\triangle$. (Bomb. Mat.)

* 1204. ABC is an isos. $\triangle$ and DE is drawn $\parallel$ to the base BC; show that $BE^2 = BC . DE + CE^2$.

* 1205. $\angle$ C is the obtuse $\angle$ of a $\triangle$ ABC and AD is $\perp$ to BC produced, and from AD produced DF is cut off = AB and DG = AC; prove that FC = GB. (Mad. Mat.)

Theorem 50.

(Euc. II. 13.)

Gen. Enun. *In any triangle, the square on the side opposite an* acute *angle is* less than *the sum of the squares on the sides containing that acute angle by twice the rectangle contained by either of these sides and the projection on it of the other.*

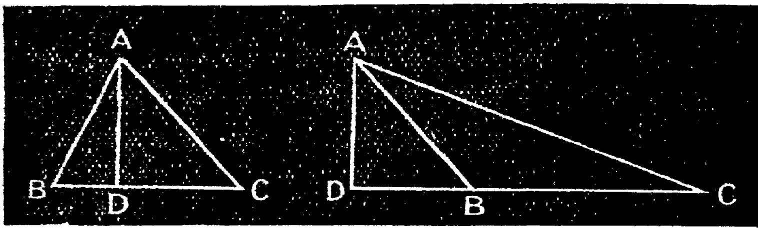


Fig. 337. Fig. 338.

Part. Enun. Let ABC be any $\triangle$ having $\angle$ C an acute $\angle$, and suppose AD drawn $\perp$ to BC (Fig. 337) or to CB produced (Fig. 338).
It is reqd. to prove that

$$AB^2 = BC^2 + CA^2 - 2BC \cdot CD.$$

Proof. $\because \angle$ BDA is a rt. $\angle$.

$\therefore AB^2 = BD^2 + DA^2$ . . . . . Th. 32.
$= (BC - CD)^2 + DA^2$ in Fig. 337.
$= (CD - BC)^2 + DA^2$ in Fig. 338.

$\therefore$ in both Fig. 337 and Fig. 338.

$AB^2 = BC^2 + CD^2 - 2BC \cdot CD + DA^2$. Th. C.

But $CD^2 + DA^2 = CA^2$ . . . . . Th. 32.

$\therefore AB^2 = BC^2 + CA^2 - 2BC \cdot CD$. Q.E.D.

Exercises.

1206. *If from an extremity of the base of an isos. $\triangle$ a $\perp$ is drawn to the opp. side, twice the rect. contained by that side and the segment adj. to the base = the sq. on the base.* (Bomb. Mat.)

1207. The side BC of an equilat. $\triangle$ ABC is produced to D and AD is joined; show that $AD^2 = BC^2 + BD^2 - BC \cdot BD$.

* 1208. If from any pt. O within a $\triangle$ ABC, $\perp$s Oa, Ob, Oc are drawn to BC, CA, AB respy., show that $AC \cdot Cb + CB \cdot Ba + BA \cdot Ac = CA \cdot Ab + AB \cdot Bc + BC \cdot Ca$. (Mad. B. C. E.)

* 1209. *The sqs. on the diags. of a trapezium are together = the sum of the sqs. on the oblique sides + twice the rect. contained by the $\parallel$ sides.* (Bomb. Sch. Fin.)

* 1210. *The sum of the sqs. on the sides of a $\square$gm = the sum of the sqs. on its diags.* (Bomb. Mat.)

* 1211. *The sum of the sqs on the diags. of any quadl. = twice the sum of the sqs. on the st. lines joining the mid. pts. of the opp sides.*

* 1212. If in any $\triangle$ ABC, $AB^2 + BC.CA = BC^2 + CA^2$, prove that $\angle$ C = 60°. (Mad. Mat.)

* 1213. $\angle$ P of the $\triangle$ PQR is an obtuse $\angle$, and QS, RT are $\perp$s to RP, QP produced; prove that $QR^2 = PQ.QT + PR.RS$. (Cal. Mat.)

* 1214. ABC is a $\triangle$, and on AB, BC, CA the sqs. ABDE, BCFG, ACHK are descd. all outside the $\triangle$; prove that $KE^2 + DG^2 + FH^2 = 3(AB^2 + BC^2 + CA^2)$. (Mad. Mat.)

* 1215. A pt. P is taken within a $\triangle$ ABC such that when $\perp$s PM, PN are let fall on AC, AB the rects. CM.AC and BN.AB are =: show that P lies on a fixed st. line $\perp$ to BC. (Bomb. Prev.)

* 1216. The base BC of a $\triangle$ ABC is divided at D so that $p.BD = q.CD$; prove that $p.AB^2 + q.AC^2 = p.BD^2 + q.CD^2 + (p+q)AD^2$.

Theorem 51.

Gen. Enun. *The sum of the squares on two sides of a triangle is double the sum of the squares on half the third side and on the median that bisects the third side.*

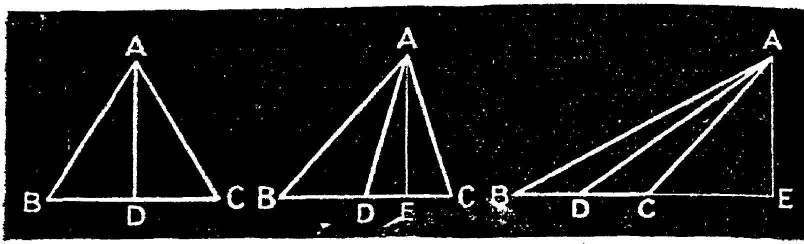


Fig. 339. Fig. 340. Fig. 341.

Part. Enun. Let AD be the median bisecting the side BC of a △ ABC.

It is reqd. to prove that

$$AB^2 + AC^2 = 2(BD^2 + AD^2).$$

Const. Suppose AE to have been drawn ⊥ to BC.

Proof. Case I. If AE coincides with AD (Fig. 339).

$AB^2 = BD^2 + AD^2$ . . . . Th. 32.

$AC^2 = CD^2 + AD^2$ . . . . Th. 32.

But BD = CD.

$\therefore BD^2 = CD^2$.

Hence, by adding,

$AB^2 + AC^2 = 2(BD^2 + AD^2)$. Q.E.D.

Case II. If AE does not coincide with AD (Figs. 340, 341).

One of the ∟s ADB, ADC must be obtuse and the other acute.

Let ∟ ADB be obtuse.

Then $AB^2 = BD^2 + AD^2 + 2 . BD . DE$. Th. 49.

And $AC^2 = CD^2 + AD^2 - 2 . CD . DE$. Th. 50.

But BD = CD.

$\therefore BD^2 = CD^2$ and $BD . DE = CD . DE$.

Hence, by adding,

$AB^2 + AC^2 = 2(BD^2 + AD^2)$. Q.E.D.

Cor. *The locus of the vertices of all triangles which have the same base and the sum of the squares on their sides equal to a given square is a circle having its centre at the middle point of the base.*

For in Figs. 340, 341,

$$AB^2 + AC^2 = 2BD^2 + 2AD^2 \quad . \quad . \quad . \quad . \quad . \quad \text{Th. 51.}$$
$$\therefore 2AD^2 = AB^2 + AC^2 - 2BD^2.$$
$$= \text{a constant.}$$

Hence **AD** is of constant length.

Exercises.

1217. *Find the locus of a pt. such that the sum of the sqs. on its distances from 2 given pts. = a given sq.* (Cal. Mat.)

1218. **P** is any pt. within a ▱gm **ABCD** whose diags. intersect at **G**; show that $PA^2 + PB^2 + PC^2 + PD^2 = AB^2 + BC^2 + 4PG^2$. (Bomb. Prev.)

1219. *Three times the sum of the sqs. on the sides of a △ = four times the sum of the sqs. on the medians.* (Cal. Mat.)

1220. **G** is the centroid of a △ **ABC**; prove that $3(GA^2 + GB^2 + GC^2) = AB^2 + BC^2 + CA^2$.

1221. *The sqs. on the sides of a quadl. are together > the sqs. on its diags. by four times the sq. on the st. line joining the mid. pts. of its diags.* (Cal. F. A.)

1222. If the sqs. on the sides of a quadl. are together = the sqs. on its diags., the quadl. is a ▱gm.

* 1223. In any quadl. if two opp. sides are bisected, the sum of the sqs. on the other two sides, together with the sqs. on the diags. = the sum of the sqs. on the bisected sides together with four times the sq. on the line joining the pts. of bisection. (Bomb. Mat.)

* 1224. **G** is the centroid of a △ **ABC** and **D** is the mid. pt. of **AG**; prove that $DB^2 + DC^2 = AB^2 + AC^2 - 10AD^2$.

* 1225. **ABCD** is a ▱gm of which **AC**, **BD** are the diags.; **P** is a pt. such that $PA^2 + PC^2 = PB^2 + PD^2$; prove that **ABCD** is a rect. (Bomb. Mat.)

* 1226. In a ⊙, **CD** is a chord ‖ to a diam. **AB**; **P** is any pt. on **AB**; show that $AP^2 + BP^2 = CP^2 + DP^2$. (Bomb. Prev.)

* 1227. **P** is any pt. on a given ⊙: **Q** is a pt. on a given concentric ⊙ such that **PQ** subtends a rt. ∟ at a fixed pt. **T**; show that the mid. pt. of **PQ** lies on a fixed ⊙. (Mad. Mat.)

RECTANGLES UNDER SEGMENTS OF CHORDS.

Theorem 52.

(Euc. III. 35, 36.)

Gen. Enun. *If two chords of a circle intersect either inside or outside the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.*

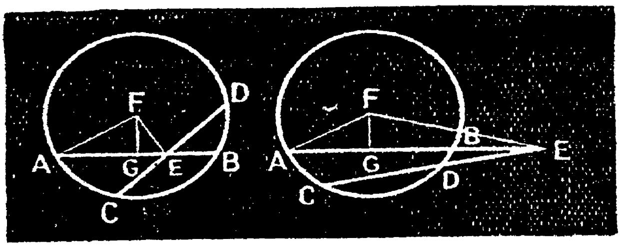


Fig. 342. Fig. 343.

Part. Enun. Let AB, CD be 2 chords of a ⊙ ABC whose centre is F and radius r linear units, intersecting at a pt. E either inside the ⊙ (Fig. 342) or outside the ⊙ (Fig. 343).

It is reqd. to prove that

$$AE \cdot EB = CE \cdot ED.$$

Const. Suppose FG to have been drawn ⊥ to AB.
Join FA, FE.

Proof. ∵ FG is ⊥ to AB . . . . . . Const.
∴ AG = BG . . . . . . Th. 35.
Hence, in Fig. 342,
$AE \cdot EB + GE^2 = AG^2$ . . . Th. D, Cor. 1.
∴ $AE \cdot EB + GE^2 + FG^2 = AG^2 + FG^2$.
∴ $AE \cdot EB = FA^2 - FE^2$ . . . Th. 32.
$= r^2$ sq. units $- FE^2$.
Similarly $CE \cdot ED = r^2$ sq. units $- FE^2$.
∴ $AE \cdot EB = CE \cdot ED$. Q.E.D.
Again, in Fig. 343,
$AE \cdot EB + AG^2 = GE^2$ . . . Th. D, Cor. 2.
∴ $AE \cdot EB + AG^2 + FG^2 = GE^2 + FG^2$.
∴ $AE \cdot EB = FE^2 - FA^2$ . . . Th. 32.
$= FE^2 - r^2$ sq. units.
Similarly $CE \cdot ED = FE^2 - r^2$ sq units.
∴ $AE \cdot EB = CE \cdot ED$ Q.E.D.

Cor. 1. *If from any point without a circle a chord and a tangent are drawn, the rectangle contained by the segments of the chord is equal to the square on the tangent.* (Proof by the Method of Limits.)

Cor. 2. *If from any point without a circle two straight lines are drawn one of which cuts the circle in two points and the other meets it, and if the rectangle contained by the whole straight line which cuts the circle and the part of it without the circle is equal to the square on the line which meets the circle, the line which meets the circle shall touch it.* (Euc. III. 37.) (Proof by Reductio ad Absurdum.)

THEOREM 53.

Gen. Enun. *If two finite straight lines intersect, or if the lines produced intersect, so that the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other, the extremities of the two lines are concyclic.*

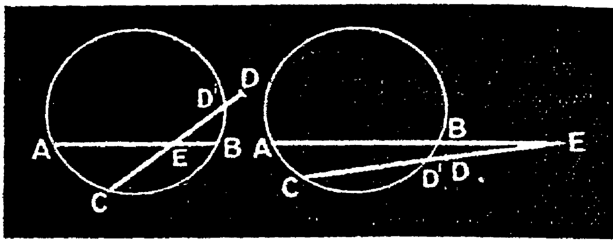


FIG. 344. FIG. 345.

Part. Enun. Let AB, CD be 2 finite st. lines which intersect, produced if necessary, at E, and let AE . EB = CE . ED.

It is reqd. to prove that

A, B, C, D *are concyclic.*

Const. Suppose the ⊙ to have been drawn that passes through A, B, C . . . . . . Th. 36.

If this ⊙ does not pass through D, let it cut CD, or CD produced, in D′.

Proof. ∵ AB, CD′ are chords of a ⊙.

∴ AE . EB = CE . ED′ . . . . . . Th. 52.

But AE . EB = CE . ED.

∴ CE . ED = CE . ED′.

∴ ED = ED′,

which is impossible.

∴ the ⊙ ABC passes through D.

∴ A, B, C, D are concyclic. Q.E.D.

Exercises.

1228. ABCD is a quadl. inscd. in a ⊙ and BA, CD produced intersect in O; prove that OA . OB = OD . OC. (Bomb. Sch. Fin.)

1229. *If 2 ⊙s intersect, their common chord when produced bisects their common tangents.* (Punj. F. E.)

1230. *If 2 $\odot$s intersect and from any pt. in the common chord produced tangents are drawn, one to each $\odot$, prove that these tangents are $=$.* (Punj. Mat.)

1231. *Find the locus of a pt. from which tangents drawn to 2 intersecting $\odot$s are $=$.* (Punj. Mat.)

* 1232. **I** is the centre of a $\odot$, **IC** a radius; from a pt. **O** without the $\odot$ **OQ** is drawn $\perp$ to **IC**. If **OC** cuts the $\odot$ at **B**, prove that **OC . CB** $=$ **2IC . CQ**. (Mad. Mat.)

* 1233. **O** is the centre of a $\odot$ and **A, B, C** are fixed pts. on its circumference; if **P** is the mid. pt. of **BC** and **AKL** any chord through **A** cutting **BC** in **K**, prove that **AK . KL** cannot be greater than $AO^2 - OP^2$. (Mad. Mat.)

* 1234. **ABC** is a $\triangle$ having $\angle$ **A** acute; prove that BC^2 is less than $AB^2 + AC^2$ by twice the sq. on the tangent drawn from **A** to the $\odot$ of which **BC** is a diam. (Bomb. Sch. Fin.)

* 1235. *Two chords of a $\odot$ are drawn at rt. $\angle$s through a fixed pt. within the $\odot$; prove that the sum of the sqs. on the chords is independent of their directions.* (Cal. F. E.)

* 1236. From a pt. **A** without a $\odot$ 2 st. lines **ABC, ADE** are drawn cutting the $\odot$ in **B, C, E, D** and a $\odot$ is descd. passing through **A, C, D** and cutting **BE** at **F**; show that **AD . AE** $= AF^2$. (Mad. F. A.)

* 1237. If **AB** is a diam. of a $\odot$ and **APQ** a st. line cutting the $\odot$ again at **P** and a fixed st. line $\perp$ to **AB** at **Q**, prove that **AP . AQ** is of constant area. (Cal. F. E.)

* 1238. The chord **AB** of a $\odot$ is produced both ways equally to **C, D** and tangents **CE, DF** are drawn on opp. sides of **CD**; show that **EF** bisects **AB**. (Cal. F. E.)

* 1239. Two $\odot$s **ABCD, EBCF**, having the common tangents **AE, DF**, cut one another at **B, C**, and the chord **BC** is produced to cut the tangents at **G, H**; show that $GH^2 = AE^2 + BC^2$. (Punj. F. A.)

PRACTICAL SECTION.

SQUARES AND RECTANGLES.

PROBLEM 24.

(Euc. II. 14.)

Gen. Enun. *To construct a square equivalent to a given rectangle.*

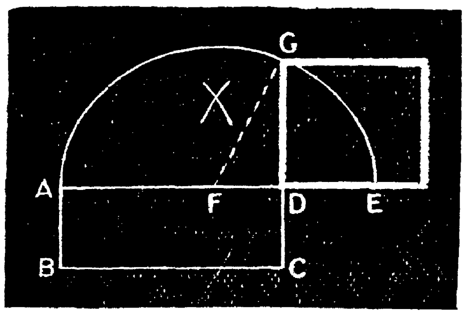


FIG. 346.

Part. Enun. Let ABCD be a given rect.
It is reqd. to construct a sq. equiv. to the rect. ABCD.

Const. Produce AD to E making DE = DC.
Bisect AE at F . . . . . . Prob. 2.
With centre F and rad. FA desc. a semi-⊙ AGE.
Produce CD to meet the semi-⊙ AGE at G.
Then shall DG^2 = rect ABCD.

Proof. Join FG.
Now $DG^2 = FG^2 - FD^2$ . . . Th. 32.
$= AF^2 - FD^2$.
$= (AF + FD)(AF - FD)$ . . Th. D.
$= AD \cdot DE$.
= rect. ABCD. Q.E.F.

Cor. *Construct a square equivalent to a given polygon.*

The Graphical Extraction of Arithmetical Square Root.

In Fig. 346 if AD measures a linear units and DE measures b linear units, then $DG^2 = ab$ square units.

$\therefore$ DG $= \sqrt{ab}$ linear units.

Thus, by giving a and b numerical values, we can find geometrically the square root of their product. For example, DG $= \sqrt{5}$ cm. if AD = 5 cm. and DE = 1 cm. Or, again, DG $= \sqrt{15}$ in. if AD = 5 in. and DE = 3 in., or if AD = 15 in. and DE = 1 in.

Exercises.

1240. Represent graphically the following lengths: (i) $\sqrt{2}$ in., (ii) $\sqrt{8}$ in., (iii) $\sqrt{\cdot 63}$ in., (iv) $\sqrt{14}$ cm., (v) $\sqrt{11}$ cm., (vi) $\sqrt{730}$ mm.

1241. Construct a sq. equiv. to any given △. (Punj. F. E.)

1242. Produce a given st. line AB to C so that rect. AB . AC = a given rect. DE . DF.

1243. Construct a rect. of given perim. equiv. to a given sq., and find when this construction fails. (Bomb. Sch. Fin.)

1244. Given a sq. and one side of a rect. equiv. to the sq., find the other side. (Allah. Mat.)

* **1245.** Construct a rect. equiv. to the difference of 2 sqs. (Cal. F. E.)

* **1246.** A st. line AB is bisected at C; find a pt. P in AB produced such that $PA^2 - PB^2 = PC^2$.

* **1247.** Divide a given st. line into 2 parts so that the rect. contained by them may be the greatest possible. (Cal. Mat.)

* **1248.** Describe an isos. obtuse ∟d △ such that the sq. on the largest side is equiv. to three times the sq. on either of the = sides. (Bomb. Mat.)

* **1249.** Find a pt. P in AB produced such that $AB^2 + AP^2 = 2AP . PB$. (Bomb. Mat.)

* **1250.** Produce a given finite st. line so that the rect. contained by the whole line thus produced and the part produced is equiv. to a given sq.

* **1251.** Divide the hypot. of a rt. ∟d △ into 2 parts such that the difference between their sqs. = the sq. on one of the sides. (Cal. Mat.)

* **1252.** Divide a given st. line internally in medial section. (Euc. II. 11.)

* **1253.** Divide a given st. line externally in medial section.

* **1254.** Produce a given finite straight line so that the sum of the sqs. on the whole line thus produced and on the part produced = 3 times the sq. on the given line. (Mad. F. A.)

* **1255.** On a given st. line as hypot. construct a rt. ∟d △ such that the sq. on one side = the rect. contained by the hypot. and the other side. (Bomb. Mat.)

Contact of Circles.

* 1256. Describe a ⊙ passing through 2 given pts. and touching a given st. line.

* 1257. Describe a ⊙ passing through 2 given pts. and touching a given ⊙.

* 1258. Describe a ⊙ passing through a given pt. and touching 2 given st. lines.

* 1259. Describe a ⊙ passing through a given pt. and touching a given st. line and a given ⊙.

PART IV.

EXPERIMENTAL SECTION.

RATIO AND PROPORTION.

Exp. 374. Draw a straight line AB and divide it into two parts

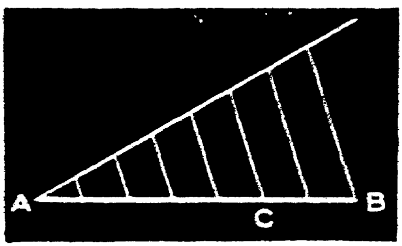


FIG. 347.

at C, so that AC = $\frac{5}{7}$ of AB.

The fraction $\frac{5}{7}$ expresses the relation which the line AC bears to the line AB in regard to quantity. This relation is called the ratio of AC to AB, and is written AC : AB. Hence the quotient $\frac{AC}{AB}$ and the ratio AC : AB are both represented by the same fraction, and one is merely another way of writing the other. And the same is true of any two magnitudes of the same kind if they are commensurable, while if they are incommensurable, they can always be expressed in terms of a common unit to any required degree of accuracy, so that we shall for our present purpose treat them as commensurable. With this understanding we shall express ratios in the quotient form when convenient. For example, the ratio $\triangle$ ABC : $\triangle$ DEF will be written in the form $\frac{\triangle ABC}{\triangle DEF}$.

Exp. 375. Measure AB and CD in inches, and work out the

FIG. 348.

ratio $\frac{AB}{CD}$ to three places of decimals. Now repeat in centimetres, and compare your results. Account for any difference there may be.

Exp. 376. Draw two straight lines AB and CD. From AB cut off a

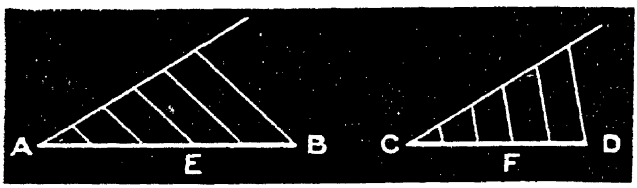


FIG. 349.

part AE, and from CD cut off a part CF such that $\frac{AE}{AB} = \frac{3}{5}$ and $\frac{CF}{CD} = \frac{3}{5}$.

Then $\frac{AE}{AB} = \frac{CF}{CD}$.

That is, the ratio of AE to AB is the same as the ratio of CF to CD
Or briefly

"AE is to AB as CF is to CD".

These four lines are said to be in proportion, and AE, AB are said to be proportional to CF, CD for AE in comparison with AB is "portion for portion the same as" CF in comparison with CD

Exp. 377. Make two triangles having unequal bases but of the same altitude. By measurement and calculation ascertain the ratio that the area of the first bears to the area of the second, and compare it with the ratio that the base of the first bears to the base of the second. Verify your conclusion theoretically.

From Exp. 377 we are led to conclude :—

The areas of triangles of equal altitude are proportional to their bases.

Learn this by heart.

Exp. 378. Make a triangle ABC having AB = 3·3 in., AC = 2·9 in. From AB cut off AD = 2·1 in. Draw DE parallel to BC, cutting AC at E. Measure AE. Work out the ratios $\frac{AD}{BD}$ and $\frac{AE}{CE}$ each to two places of decimals, and compare.

Exp. 379. Repeat Exp. 378, varying the length of AD.

From Exps. 378, 379, allowing for errors of measurement, we are led to conclude :—

If a straight line is drawn parallel to one side of a triangle, the other two sides are divided in the same ratio.

Learn this by heart.

Exp. 380. Make a triangle ABC, and divide its sides AB and AC at D and E respectively, so that $\frac{AD}{BD} = \frac{AE}{CE}$. (This may be done by taking AD any fraction, say ·7, of AB and AE the same fraction of AC.) Join DE and verify the following truth by testing with ruler and set square :—

If a straight line divides two sides of a triangle in the same ratio, it is parallel to the third side.

Learn this by heart.

Exp. 381. ABC is a triangle, and PQ is drawn parallel to BC, cutting AB at P, and AC at Q. If AB = 3·8 in., AC = 2·4 in., AP = 2·4 in., calculate the length of AQ, and verify by drawing and measurement.

Exp. 382. ABC is a triangle, and PQ is drawn parallel to BC, cutting AB at P, and AC at Q. If AB = 9·3 cm., AC = 6·9 cm., BP = 4·2 cm., calculate the length of AQ, and verify by drawing and measurement.

Exp. 383. ABCD is a quadrilateral, EF is drawn parallel to AD, cutting AB at E, and BD at F. FG is drawn parallel to DC, cutting BC at G. Then AB, BC are cut in the same ratio at E and G (why ?). If AB = 3·25 in., BC = 2·88 in., AE = 1·1 in., calculate the length of BG and verify by drawing and measurement.

Exp. 384. Make a triangle ABC having $\angle A = 76°$ and $\angle B = 57°$. Make another triangle DEF having $\angle D = \angle A$ and $\angle E = \angle B$ but the side DE not necessarily equal to the side AB. Then the triangle DEF and the triangle ABC are equiangular (why ?) and the sides that are opposite to equal angles are called corresponding sides. Now measure and work out the ratios $\frac{AB}{DE}$, $\frac{BC}{EF}$, $\frac{CA}{FD}$ each to two places of decimals and compare the results.

Exp. 385. Repeat Exp. 384 varying the size of the angles and the length of the sides. 12 *

From Exps. 384, 385, allowing for errors of measurement, we are led to conclude :—

If two triangles are equiangular, each side of the first bears the same ratio to the corresponding side of the second.

Learn this by heart.

Triangles that are equiangular and have therefore each side of the one bearing the same ratio to the corresponding side of the other are said to be similar.

Exp. 386. Make two triangles ABC, DEF such that AB bears the same ratio to DE as BC to EF and as CA to FD. (This is done by taking DE any fraction, say ·8, of AB, EF the same fraction of BC and FD the same fraction of CA.) Now verify the following truth by measurement with protractor :—

If each side of one triangle bears the same ratio to each side of another triangle, taken in order, the triangles are equiangular.

Learn this by heart.

Exp. 387. ABC is a triangle and PQ is drawn parallel to BC cutting AB at P and AC at Q. If AB = 3·1 in., BC = 4·2 in., AP = 1·9 in., calculate the length of PQ and verify by drawing and measurement.

Exp. 388. ABC is a triangle right angled at A and AD is the perpendicular from A to BC. Then the triangles ABD, ACD, ABC are similar (why?). If AB = 9·3 cm., BC = 10·8 cm., CA = 5·5 cm., calculate the lengths of BD, CD, AD and verify by drawing and measurement.

Exp. 389. ABC is a triangle. A circle touches BC at C, passes through A and cuts BA produced at D. Then the triangles ABC, BCD are similar (why?). If AB = 6·8 cm., BC = 7·9 cm., CA = 5 cm., calculate the lengths of CD and AD and verify by drawing and measurement.

Exp. 390. ABC is the circum-circle of a triangle ABC. Another circle ADC touches AB at A and passes through C. The tangent to the circle ABC at A cuts the circle ACD at D. Then the triangles ABC, ACD are similar (why?). If AC = 4·8 cm., BC = 5·9 cm., AB = 6·6 cm., calculate the lengths of CD and AD and verify by drawing and measurement.

Exp. 391. Make a triangle ABC having $\angle A = 56°$, AB = 2·5 in., AC = 2·9 in. Make another triangle DEF having $\angle D = \angle A$ and $\frac{DE}{DF} = \frac{AB}{AC}$ (say DE = 1·9 in.). Now measure BC and EF and show that the triangles ABC and DEF have each side of the one bearing the same ratio to each side of the other taken in order and are therefore similar.

From Exp. 391 we are led to conclude :—

If the ratio of two sides of one triangle be equal to the ratio of two sides of another triangle, and if the angles contained by these sides be equal the triangles are similar.

Learn this by heart

Exp. 392. Make a triangle ABC having AB = 8·3 cm., BC = 7 cm., CA = 12·1 cm. From AB cut off a part AD = 6 cm., and from AC cut off a part AE such that $\frac{AB}{AC} = \frac{AD}{AE}$. Calculate the length of DE, and verify by drawing and measurement.

Exp. 393. Make a triangle ABC having AB = 7·7 cm., BC = 4·1 cm., CA = 5·2 cm. Produce BC to D so that CD = BC. Join AD. Make another triangle EFG similar to the triangle ABD, but having FG = 5·2 cm. Now calculate the length of the median of the triangle EFG through E and verify by drawing and measurement.

Exp. 394. Make a triangle ABC having AB = 2·7 in., BC = 3·6 in., CA = 2·3 in. Draw AD bisecting $\angle$A *internally* and cutting BC at D. Measure BD and CD and work out the ratios $\frac{BD}{CD}$ and $\frac{AB}{AC}$ each to two places of decimals and compare the results.

Exp. 395. Repeat Exp. 394 varying the measurements.

Exp. 396. Make a triangle ABC having AB = 4·4 in., BC = 2·1 in., CA = 2·6 in. Draw AD bisecting $\angle$A *externally* and cutting BC produced at D. Measure BD and CD and work out the ratios $\frac{BD}{CD}$ and $\frac{AB}{AC}$ each to two places of decimals and compare the results.

Exp. 397. Repeat Exp. 396 varying the measurements.

From Exps. 394 to 397, allowing for errors of measurement, we are led to conclude :—

The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle, and likewise the external bisector externally.

Learn this by heart.

Exp. 398. Make a triangle ABC having AB = 8·6 cm., BC = 9·2 cm., CA = 6·7 cm. Draw AD bisecting $\angle$A internally and cutting BC at D. Calculate the lengths of BD and CD and verify by measurement.

Exp. 399. Make a triangle ABC having AB = 3·6 in., BC = 2·2 in , CA = 1·7 in. Draw AD bisecting $\angle$A externally and cutting BC produced at D. Calculate the lengths of BD and CD and verify by measurement.

Exp. 400. Make a triangle ABC having AB = 8·5 cm., BC = 8·8 cm., CA = 6·8 cm. Draw AD bisecting $\angle$A internally and BE bisecting $\angle$B internally. Let AD and BE intersect at I. Now work out the ratio $\frac{AI}{DI}$ to two places of decimals and verify by measurement. (See Properties of Proportion IX., page 355.)

Exp. 401. Make a triangle ABC having base BC = 3·2 in. and altitude AP = 2·4 in. Make another triangle DEF equiangular and therefore similar to the triangle ABC but let its base EF = 2·8 in. Then the altitude of the triangle DEF and AP are proportional to EF and BC (why?). Now evaluate and compare the ratio

$$\frac{\text{area of triangle ABC}}{\text{area of triangle DEF}}$$

with the ratio

$$\frac{\text{area of square on BC}}{\text{area of square on EF}}.$$

Exp. 402. Repeat Exp. 401 varying the measurements.

From Exps. 401, 402 we are led to conclude :—

The areas of similar triangles are to one another as the squares on corresponding sides.

Learn this by heart.

Exp. 403. Make a triangle ABC having BC = 8·5 cm. and area = 27·2 sq. cm. and from it cut off a triangle ADE by a straight line DE parallel to BC such that the triangle ADE is $\frac{9}{16}$ of the triangle ABC. Verify by measuring the base and altitude of the triangle ADE and calculation.

Exp. 404. Make a triangle ABC having AB = 5·3 cm., AC = 3·1 cm., BC = 5 cm. Draw the tangent at A to the circumcircle of the triangle ABC and let it meet BC produced at D. Then $\triangle$ABD and $\triangle$ACD are similar (why?). Evaluate the ratio $\frac{\triangle ABD}{\triangle ACD}$ to three places of decimals and verify by measuring CD and using the result of Exp. 377.

THEORETICAL SECTION.

RATIO.

The quotient of one magnitude by another expresses the ratio of the first magnitude to the second. Thus if one magnitude A contains a particular unit a times, and another magnitude B contains the same unit b times, the ratio of magnitude A to magnitude B is expressed by the quotient of A by B, which in this case is equal to the fraction $\frac{a}{b}$.

Euclid defines Ratio as " the mutual relation of two magnitudes of the same kind to one another in respect of quantity ". This definition, however, cannot be accepted as satisfactory, for the idea of ratio is a fundamental conception, and cannot be precisely defined.

If A and B are incommensurable magnitudes, the quotient of A by B can only approximately be expressed numerically. Since, however, the measures of incommensurable magnitudes of the same kind can be expressed in terms of any chosen unit to as many decimal places as we please, we shall, for our present purpose, always regard them as commensurable.

The ratio of A to B is written A : B. It is also put in the quotient forms $\frac{A}{B}$, A/B and A ÷ B.

Of the terms of the ratio $\frac{A}{B}$, the numerator A is called the antecedent and the denominator B is called the consequent of the ratio.

Note 1. We can only speak of the ratio of one magnitude to another *of the same kind* just as we can only speak of the quotient of one magnitude by another of the same kind. For example, we can speak of the ratio of one area to another area or of one sum of money to another sum of money, but not of an area to a length or of a weight to an interval of time.

Note 2. The ratio of one magnitude to another is an *abstract number* and is independent of the particular unit or units in terms of which the magnitudes are expressed. Thus the ratio of 7 inches to 15 inches is $\frac{7}{15}$ and not $\frac{7}{15}$ inches.

The ratio $\frac{A}{B}$ is said to be one of

greater inequality if A is greater than B,
equality if A is equal to B,
less inequality if A is less than B.

Exercises.

1260. Express the following theorems algebraically and supply a proof in each case:—

(a) *A ratio is unaltered by multiplying antecedent and consequent by the same number.*

(b) A ratio of greater inequality is diminished by adding the same quantity to antecedent and consequent.

(c) A ratio of less inequality is increased by adding the same quantity to antecedent and consequent.

(d) A ratio of greater inequality is increased by subtracting the same quantity from antecedent and consequent.

(e) A ratio of less inequality is diminished by subtracting the same quantity from antecedent and consequent.

1261. Find the ratio of 1000 sq. in. to 5 sq. ft.

1262. Express the ratio of 13 cm. to 27 cm. as a decimal fraction to 3 places.

1263. Write down the ratio of the diagonal of any square to its side and verify by construction and proof.

1264. Write down the ratio of the altitude of any equilateral triangle to its side and verify by construction and proof.

1265. Which is the greater ratio 32 yds. to 55 yds. or 37 in. to 60 in.?

1266. What quantity added to the antecedent and consequent of the ratio 5 ft. : 9 ft. will convert it into the ratio 11 : 13?

1267. ABC is a right-angled $\triangle$. The ratio of side AB to side BC is $\frac{4}{5}$. The square on hypotenuse AC measures 164 sq. in. Find AB and BC.

1268. ABC is a right-angled $\triangle$. The ratio of hypotenuse AB to side BC is $\frac{3}{2}$. The square on side AC measures 125 sq. cm. Find AB and BC.

1269. Divide 238° into two parts in the ratio of 5 to 9.

1270. If $2x^2 - 5xy - 3y^2 = 0$, find the ratio of x to y.

PROPORTION.

Def. 68. If the ratio of one magnitude to a second is equal to the ratio of a third to a fourth, the four magnitudes are said to be in **proportion**, and the first and second are said to be **proportional** to the third and fourth.

Thus if A, B, C, D are four magnitudes, and $\frac{A}{B} = \frac{C}{D}$ or $A : B = C : D$, we say that A and B are proportional to C and D, or "A is to B as C is to D". This proportion is sometimes written $A : B :: C : D$.

In any proportion $\frac{A}{B} = \frac{C}{D}$ the terms A, D are called the **extremes** and B, C the **means**; also D is called the **fourth proportional** to A, B and C.

Note.—When four magnitudes are in proportion, the two compared in the first ratio need not be of the same kind as the two compared in the second ratio. This is implied in Def. 68.

$$\text{For example, } \frac{2 \text{ ft. } 4 \text{ in.}}{3 \text{ ft. } 6 \text{ in.}} = \frac{1 \text{ ton } 2 \text{ cwt.}}{1 \text{ ton } 13 \text{ cwt.}}$$

Since each ratio $= \frac{2}{3}$.

Def. 69. If the ratio of one magnitude to a second is equal to the ratio of the second to a third, the three magnitudes are said to be **in proportion.**

Thus if A, B, C are three magnitudes in proportion, we have $\frac{A}{B} = \frac{B}{C}$.

In this proportion the term B is called the **mean proportional** between A and C; also C is called the **third proportional** to A and B.

Note.—When three magnitudes are in proportion, they must all be of the same kind. This is implied in Def. 69.

Def. 70. If $\frac{A}{B} = \frac{B}{C} = \frac{C}{D} =$ etc., the magnitudes A, B, C, D, etc., are said to be in **continued proportion**

PROPERTIES OF PROPORTION.

I. If $\frac{A}{B} = \frac{C}{D}$, then $\frac{B}{A} = \frac{D}{C}$ (Invertendo)

$$\text{for } \frac{B}{A} = 1 \div \frac{A}{B} = 1 \div \frac{C}{D} = \frac{D}{C}.$$

II. If $\frac{A}{B} = \frac{C}{D}$, then $\frac{A+B}{B} = \frac{C+D}{D}$ (Componendo)

$$\text{for } \frac{A}{B} + 1 = \frac{C}{D} + 1$$
$$\therefore \frac{A+B}{B} = \frac{C+D}{D}.$$

III. If $\frac{A}{B} = \frac{C}{D}$, then $\frac{A-B}{B} = \frac{C-D}{D}$ (Dividendo)

$$\text{for } \frac{A}{B} - 1 = \frac{C}{D} - 1$$
$$\therefore \frac{A-B}{B} = \frac{C-D}{D}.$$

IV. If $\frac{A}{B} = \frac{C}{D}$, then $\frac{A+B}{A-B} = \frac{C+D}{C-D}$ (Componendo and Dividendo)

$$\text{for } \frac{A+B}{A-B} = \frac{\frac{A}{B}+1}{\frac{A}{B}-1} = \frac{\frac{C}{D}+1}{\frac{C}{D}-1} = \frac{C+D}{C-D}.$$

V. If $\frac{A}{B} = \frac{P}{Q}$ and $\frac{B}{C} = \frac{Q}{R}$, then $\frac{A}{C} = \frac{P}{R}$ (Ex aequali)

$$\text{for } \frac{A}{B} \times \frac{B}{C} = \frac{P}{Q} \times \frac{Q}{R}$$
$$\therefore \frac{A}{C} = \frac{P}{R}.$$

VI. If $\frac{a}{b} = \frac{c}{d}$ where a, b, c, d are abstract numbers, then $\frac{a}{c} = \frac{b}{d}$ (Alternando)

$$\text{for } \frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}$$
$$\therefore \frac{a}{c} = \frac{b}{d}.$$

VII. If $\frac{a}{b}=\frac{c}{d}$ where a, b, c, d are abstract numbers, then $ad = bc$

$$\text{for } \frac{a}{b} \times bd = \frac{c}{d} \times bd$$
$$\therefore ad = bc.$$

Cor. 1. If $\frac{a}{b}=\frac{b}{c}$ where a, b, c are abstract numbers, then $ac = b^2$.

Cor. 2. *If four straight lines are in proportion, the rectangle contained by the extremes is equal to the rectangle contained by the means.* (Euc. VI. 16.)

Cor. 3. *If three straight lines are in proportion, the rectangle contained by the extremes is equal to the square on the mean.* (Euc. VI. 17.)

VIII. If $ad = bc$, where a, b, c, d are abstract numbers, then $\frac{a}{b}=\frac{c}{d}$,

$$\text{for } ad \div bd = bc \div bd$$
$$\therefore \frac{a}{b}=\frac{c}{d}.$$

Cor. 1. If $ac = b^2$ where a, b, c are abstract numbers, then $\frac{a}{b}=\frac{b}{c}$.

Cor. 2. *If the rectangle contained by one pair of straight lines is equal to the rectangle contained by another pair, the four straight lines are in proportion.* (Euc. VI. 16.)

Cor. 3. *If the rectangle contained by two straight lines is equal to the square on a third, the three straight lines are in proportion.* (Euc. VI. 17.)

IX. If $\frac{a}{p}=\frac{b}{q}=\frac{c}{r}= \ldots$ where $a, b, c \ldots p, q, r \ldots$ are abstract numbers, then $\frac{a}{p}=\frac{a+b+c+\ldots}{p+q+r+\ldots}$

$$\text{for if } \frac{a}{p}=\frac{b}{q}=\frac{c}{r}=\ldots=k$$
$$\text{then } a = pk,\ b = qk,\ c = rk \ldots$$
$$\therefore a+b+c+\ldots = k(p+q+r+\ldots)$$
$$\therefore \frac{a+b+c+\ldots}{p+q+r+\ldots} = k = \frac{a}{p}.$$

Exercises.

1271. Are the following statements correct:—

$$5 \text{ in.} : 7 \text{ in.} = 5 \text{ sq. in.} : 7 \text{ sq. in.}$$
$$5 \text{ in.} : 5 \text{ sq. in.} = 7 \text{ in.} : 7 \text{ sq. in.}?$$

1272. Complete the proportion $2\frac{3}{4}$ yds. : $5\frac{1}{2}$ yds. = $1\frac{1}{4}$ oz. :

1273. Find the fourth proportional to ·15, 5 and 4·5.

1274. Find the fourth proportional to 5, ·15 and 4·5.

1275. Find the third proportional to 1 sq. ft. and 36 sq. in.

1276. Find the third proportional to 36 sq. in. and 1 sq. ft.

1277. Find the mean proportional between 4 shillings and 3 pence.

1278. Find the mean proportional between 3 pence and 4 shillings.

1279. Find the third proportional to $1 + \sqrt{2}$ and $3 + 2\sqrt{2}$.

1280. A mixture contains 32 parts gold, 5 parts silver and 3 parts copper; what is the percentage of the copper in the mixture?

1281. Prove that the fourth proportional of a, b, b is the third proportional of a, b.

1282. *If a, b, c are abstract numbers, and $a : b = b : c$, prove that $a : c = a^2 : b^2 = b^2 : c^2$ and $b = \sqrt{ac}$.*

1283. If a, b, c, d are abstract numbers, and $a : b = b : c = c : d$, prove that $a : d = a^3 : b^3 = b^3 : c^3 = c^3 : d^3$ and $b = \sqrt[3]{a^2 d}$, $c = \sqrt[3]{ad^2}$.

1284. If $a : b = c : d$, and b is a mean proportional between c and d, prove that c is a mean proportional between a and b.

1285. *Prove that a given straight line can be divided internally in a given ratio at one and only one point.*

1286. *Prove that a given straight line can be divided externally in a given ratio at one and only one point.*

1287. Prove that the rectangle contained between two st. lines is a mean proportional between the squares on the lines.

PROPORTIONAL DIVISION OF STRAIGHT LINES. SIMILAR TRIANGLES.

THEOREM 54.

(Euc. VI. 2.)

Gen. Enun. (A) *If a straight line is drawn parallel to one side of a triangle the other two sides are divided in the same ratio.*

Conversely

(B) *If a straight line divides two sides of a triangle in the same ratio, it is parallel to the third side.*

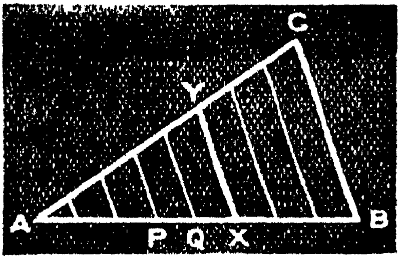


FIG. 350.

Part. Enun. (A) Let the st. line XY be drawn ∥ to the side BC of a △ ABC cutting AB and AC at X and Y respy.

It is reqd. to prove that

$$\frac{AX}{BX} = \frac{AY}{CY}.$$

Const. and Proof. Suppose AX and BX to have a common measure PQ and suppose

$$AX = m \,.\, PQ$$
$$\text{and } BX = n \,.\, PQ$$
$$\text{so that } \frac{AX}{BX} = \frac{m}{n}.$$

Suppose AB divided into $m + n$ equal parts.

Then AX will contain m and BX will contain n of these parts.

And each part = PQ.

Through the pts. of division in AB suppose st. lines drawn ∥ to BC.

These parallels will divide AC into $m + n$ equal parts, of which AY will contain m and CY will contain n . . Th. 23.

$$\text{so that } \frac{AY}{CY} = \frac{m}{n}$$
$$\therefore \frac{AX}{BX} = \frac{AY}{CY}. \qquad \text{Q.E.D.}$$

Part. Enun. (B) Let the st. line XY divide the sides AB, AC of a $\triangle$ ABC at X, Y so that $\frac{AX}{BX} = \frac{AY}{CY}$.

It is reqd. to prove that

XY *is* $\parallel$ *to* BC.

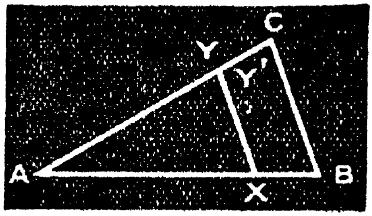


FIG. 351.

Proof. If XY is not $\parallel$ to BC suppose the $\parallel$ to BC through X to cut AC at Y'.

Then $\frac{AX}{BX} = \frac{AY'}{CY'}$ . . Th. 54 (A).

But $\frac{AX}{BX} = \frac{AY}{CY}$ . . . Hyp.

$\therefore \frac{AY}{CY} = \frac{AY'}{CY'}$

$\therefore \frac{AY + CY}{CY} = \frac{AY' + CY'}{CY'}$. . . . Componendo.

that is $\frac{AC}{CY} = \frac{AC}{CY'}$.

$\therefore$ Y and Y' coincide.

$\therefore$ XY is parallel to BC.

Q.E.D.

Similarly if XY be drawn $\parallel$ to BC cutting AB and AC produced either way

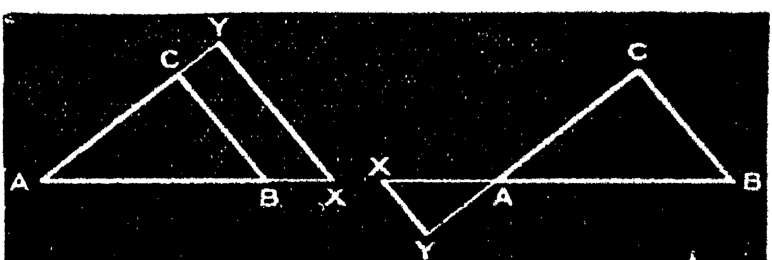


FIG. 352. FIG. 353.

at X and Y respy. it can be proved that

$$\frac{AX}{BX} = \frac{AY}{CY}.$$

And, conversely,
if XY divides the sides AB, AC of a $\triangle$ ABC produced either way at X, Y so that $\frac{AX}{BX} = \frac{AY}{CY}$ it can be proved that

XY is $\parallel$ to BC.

Cor. *If a st. line* XY *be drawn* $\parallel$ *to the side* BC *of a* $\triangle$ ABC *cutting* AB *and* AC *at* X, Y *then* $\frac{AB}{AX} = \frac{AC}{AY}$ *and* $\frac{AB}{BX} = \frac{AC}{CY}$ *and conversely.*

Exercises.

1288. *The st. line joining the mid. pts. of the sides of a* $\triangle$ *is* $\parallel$ *to the base.*

1289. ABCD is a quadl. having AB $\parallel$ to CD; prove that the st. line joining the mid. pts. of AD and BC is $\parallel$ to AB.

1290. ABCD *is a quadl. having* AB $\parallel$ *to* CD; *prove that any st. line drawn* $\parallel$ *to* AB *cuts* AD *and* BC *proportionally.*

1291. ABCD is a quadl. having AB $\parallel$ to CD; prove that any st. line cutting AD and BC proportionally is $\parallel$ to AB.

1292. Points D, E, F are taken on the sides AB, AC, BC respy. of a $\triangle$ ABC such that DE is $\parallel$ to BC and EF is $\parallel$ to AB; prove that $\frac{AB}{AD} = \frac{BC}{BF}$.

1293. *The diags. of a trapezium cut one another proportionally.*

1294. E is a pt. in the side AB of a quadl. ABCD; EF $\parallel$ to AD meets BD in F and EG $\parallel$ to AC meets BC in G. Prove that FG is $\parallel$ to CD.

1295. E is a pt. in the diag. BD of a quadl. ABCD; EF $\parallel$ to DA meets AB in F and EG $\parallel$ to DC meets CB in G. Prove that FG is $\parallel$ to AC.

1296. *If* 2 *st. lines are cut by* 3 *or more* $\parallel$ *st. lines, the intercepts on the one have the same ratios as the intercepts on the other.*

1297. O is a fixed pt. and PQ a fixed st. line; OR is any st. line drawn from O to PQ and S is a pt. in OR such that the ratio $\frac{OS}{OR}$ is const. Find the locus of S.

1298. BE bisecting the median through A of a $\triangle$ ABC meets AC at E. Prove that E is a pt. of trisection of AC.

* 1299. *The areas of* $\triangle$*s and* ▱*gms of the same altitude are to one another as their bases.* (Euc. VI. 1.)

* 1300. *If* ABC, PQR *are* 2 $\triangle$*s of = area having* $\angle A = \angle P$ *prove that* $\frac{AB}{PR} = \frac{PQ}{AC}$; *and conversely if* $\angle A = \angle P$ *and* $\frac{AB}{PR} = \frac{PQ}{AC}$ *prove that* $\triangle ABC = \triangle PQR$ (Euc. VI. 15.)

* 1301. *If* ABCD, PQRS *are* 2 ▱*gms of = area having* $\angle B = \angle S$ *prove that* $\frac{AB}{SR} = \frac{PS}{BC}$ *and conversely if* $\angle B = \angle S$ *and* $\frac{AB}{SR} = \frac{PS}{BC}$ *prove that* ▱*gm.* ABCD = ▱*gm.* PQRS. (Euc. VI. 14.)

* 1302 ST is drawn ‖ to the side QR of a $\triangle$ PQR meeting PQ at S and PR at T. If RS, QT intersect at V prove that PV lies along a median of the $\triangle$ PQR.

* 1303. XY is drawn parallel to BC the base of a $\triangle$ ABC cutting CA produced and BA produced at X, Y respy.; if D, E are the mid. pts. of BY, CX prove that DE is ‖ to BC.

* 1304. A st. line is drawn cutting the sides BC, CA, AB (produced if necessary) of a $\triangle$ ABC in D, E, F respy., and it is equally inclined to AC and AB; prove that $\frac{BD}{CD} = \frac{BF}{CE}$.

* 1305. PQR is a $\triangle$; PS is the $\perp$ from P to the internal bisector of $\angle Q$; ST drawn ‖ to QR meets PR in T. Prove that T is the mid. pt. of PR.

* 1306. QS, RT are the bisectors of the external $\angle$s at the base QR of a $\triangle$PQR and S, T are the feet of the $\perp$s from P on QS, RT; prove that ST is ‖ to QR.

* 1307. If the line joining a fixed pt. to any pt. on a fixed ⊙ be divided at P in a constant ratio, find the locus of P.

Theorem 55.

(Euc. VI. 4.)

Note.—The sides of equiangular triangles, which are opposite to equal angles, are called corresponding sides.

Gen. Enun. *If two triangles are equiangular each side of the first bears the same ratio to the corresponding side of the second.*

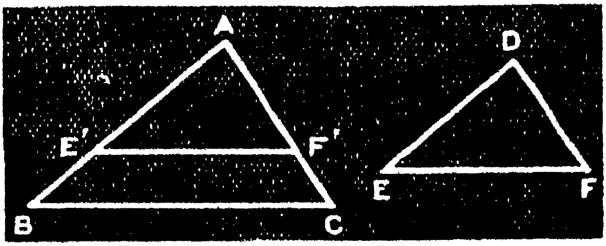


Fig. 354.

Part. Enun. Let ABC, DEF be 2 △s, having

$$\begin{cases} \angle A = \angle D \\ \angle B = \angle E. \\ \angle C = \angle F \end{cases}$$

It is reqd. to prove that

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}.$$

Const. $\because \angle A = \angle D$.

△ DEF can be placed so that pt. D coincides with pt. A, side DE falls along side AB and side DF along side AC.

Let E′, F′ be the new positions of E, F.

Proof. $\because \angle AE'F' = \angle E$ . . . . . . Const.
$= \angle B$ . . . . . . Hyp.
$\therefore$ E′F′ is $\parallel$ to BC . . . . . . Th. 5.

And $\therefore \dfrac{AB}{AE'} = \dfrac{AC}{AF'}$ . . . . Th. 54 Cor.

that is $\dfrac{AB}{DE} = \dfrac{AC}{DF}$.

Similarly by placing △ DEF so that pt. E coincides with pt. B, side ED falls along side BA and side EF along side BC, it can be shown that

$$\frac{AB}{DE} = \frac{BC}{EF}.$$

$$\therefore \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}.$$

Q. E. D.

Theorem 56.

(Euc. VI. 5.)

Gen. Enun. *If each side of one triangle bears the same ratio to each side of another, taken in order, the triangles are equiangular.*

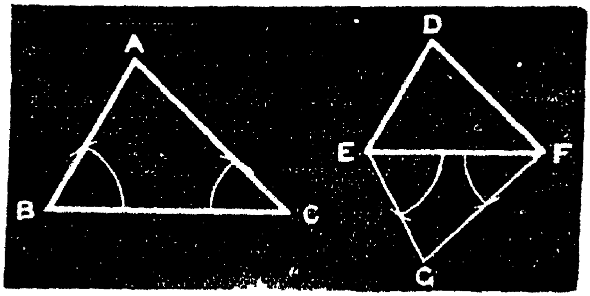


Fig. 355.

Part. Enun. Let ABC, DEF be 2 $\triangle$s in which

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}.$$

It is reqd. to prove that

$\triangle$*s* ABC, DEF *are equiang.*

Const. Suppose $\angle$ FEG = $\angle$ B and $\angle$ EFG = $\angle$ C, the pts. D and G being on opp. sides of EF.

Proof. $\because$ $\triangle$s ABC, GEF are equiang. . . Th 8, Cor. 2.

$\therefore \dfrac{AB}{GE} = \dfrac{BC}{EF}$ . . . . . Th. 55.

$= \dfrac{AB}{DE}$ . . . . . Hyp.

$\therefore$ GE = DE.

Similarly GF = DF.

Also EF is common to $\triangle$s GEF, DEF.

Hence $\triangle$GEF $\equiv$ $\triangle$DEF . . . . . . Th. 14.

$\therefore$ $\triangle$s GEF, DEF are equiang.

But $\triangle$s GEF, ABC ,, ,, ... Const.

$\therefore$ $\triangle$s ABC, DEF ,, ,,

Q.E.D.

Note.—Triangles which are equiangular, and have therefore each side of the one bearing the same ratio to the corresponding side of the other, are said to be similar.

Exercises.

1308. *The st. line joining the mid. pts. of the sides of a △ is ½ the base.*

1309. *Every st. line drawn ∥ to a side of a △ cuts off a similar △.*

1310. BE is the internal bisector of ∠B of a △ ABC and D, E are the feet of the ⊥s from A, C on BE; prove that AD . BE = BD . CE.

1311. *In similar △s corresponding sides are proportional to corresponding altitudes.*

1312. If one of the ∥ sides of a trapezium is double the other prove that the diagonals meet in a pt. of trisection.

1313. *Deduce from Ex. 1312 that the medians of a △ trisect each other.*

1314. *If 2 ∥ st. lines are cut by 3 or more st. lines drawn from a common pt., prove that the intercepts on the one are proportional to the corresponding intercepts on the other.*

1315. A, B, C are fixed pts. in a st. line, and CD is any st. line through C; prove that the ⊥s from A and B on to CD bear to one another a constant ratio.

1316. D, E are the mid. pts. of the sides AB, BC of a △ ABC and DF, EG are drawn ∥ to one another, meeting AC in F, G; prove that DF = EG.

1317. *If from any pt. without a ⊙ a chd. and a tangent are drawn, prove that the rect. contained by the segments of the chd. = the sq. on the tangent.*

1318. *If 2 chds. of a ⊙ intersect either inside or outside the ⊙, the rect. contained by the segments of the one = the rect. contained by the segments of the other.*

1319. ABCD is a ▱gm and BEF is drawn meeting AD produced at F and cutting CD at E; prove that $\frac{AF}{BF} = \frac{BC}{BE}$.

1320. *If α be an angle whose vertex is O and from any pt. P in either of the arms a ⊥ PN be drawn to the other arm, prove that the ratio* $\frac{PN}{OP}$ *is the same for all positions of P.*

Note.—The ratio $\frac{PN}{OP}$ is called the sine of the angle α (written sin α).

1321. *In the fig. of Ex. 1320 prove that the ratio* $\frac{ON}{OP}$ *is the same for all positions of P.*

Note.—The ratio $\frac{ON}{OP}$ is called the cosine of the angle α (written cos α).

1322. *In the fig. of Ex. 1320 prove that the ratio* $\frac{PN}{ON}$ *is the same for all positions of P.*

Note.—The ratio $\frac{PN}{ON}$ is called the tangent of the angle a (written tan a).

1323. *In a rt.* ∟*d* △ ABC *a* ⊥ AD *is drawn from the rt.* ∟A *to the hypot.* BC. *Prove that* (1) AD *is the mean proportional between* BD *and* CD, (2) AB *is the mean proportional between* BC *and* BD.

1324. PQ is a fixed diameter of a ⊙ and S is a pt. on the tangent at Q; if PS cuts the circumfce. at R prove that the rect. PR . PS is constant for all positions of S.

1325. PQR is a △ and VPX is drawn through P ∥ to QR; if PX = PV and QX cuts PR at S and RV cuts PQ at T prove that TS is ∥ to QR.

1326. PQRS is a quadl. having PS ∥ to QR; PR and QS meet at T and XTV is drawn ∥ to PS meeting PQ and SR at X and V respy. Prove that XT = VT.

1327. If 2 △s are on = bases and between the same parallels, any st. line ∥ to their bases will cut off equal areas from the △s.

1328. The tangent to a ⊙ at P cuts 2 ∥ tangents at Q, R whose pts. of contact are S, T respy. If QT, RS intersect at X prove that PX is ∥ to QS and RT.

1329. The corresponding sides of similar △s are to one another as their circum-radii.

1330. P and Q are fixed pts. and AB is any st. line passing between them; prove that if the ⊥s from P and Q on to AB bear a constant ratio to one another, AB must pass through a fixed pt.

1331. A ⊙ of centre P touches a ⊙ of centre Q externally at R and a st. line SRT is drawn meeting the first ⊙ at S and the second ⊙ at T; prove that PR . RT = QR . RS.

*1332. D, E, F are pts. on the sides BC, CA, AB of a △ ABC such that DE is ∥ to BA and EF is ∥ to BC. Prove that △ DEF is a mean proportional to △s AFE and DEC.

*1333. If P, Q are pts. of contact of a direct common tangent to 2 ⊙s that touch one another externally, prove that PQ is a mean proportional between the diams. of the ⊙s.

*1334. If P, Q are pts. of contact of a direct common tangent to 2 ⊙s that touch one another externally at T and if PQ meets the line of centres in R, prove that TR is the mean proportional between PR and QR.

*1335. The tangent at P to a fixed ⊙ meets 2 ∥ tangents at Q and R; prove that rect. PQ . PR is const.

*1336. ABC is a △ and D, E are pts. in AB, AC such that BD = CE. If DE, BC produced meet at F, prove that $\frac{AB}{AC} = \frac{EF}{DF}$.

*1337. AB is a diam. of a ⊙ and BEF is the tangent at B; AE is drawn to cut the circumfce. at C and AF to cut the circumfce. at D. Prove that △ ACD is similar to △ AEF.

* 1338. *If 2 $\triangle$s have one $\angle$ of the one = one $\angle$ of the other, and the sides about 2 other $\angle$s in proportion, then the remaining $\angle$s are either = or supplementary.* (Euc. VI. 7.)

* 1339. **PQ** is one of the = sides of an isos. $\triangle$ **PQR**; **ST** bisecting **PQ** at rt. $\angle$s meets the base **QR** produced at **T**. Prove that $PQ^2 = QR \cdot QT$.

* 1340. The internal bisector of the vert. $\angle$ **A** of a $\triangle$ **ABC** meets the base **BC** at **D** and the circum-$\odot$ at **E**. Prove that $AB \cdot AC = AD \cdot AE$.

* 1341. *In the fig. of Ex.* 1840 *prove that* $AB \cdot AC = BD \cdot DC + AD^2$. (Euc. VI. B.)

* 1342. *The external bisector of the vert.* $\angle$ **A** *of a* $\triangle$ **ABC** *meets the base* **BC** *produced at* **D**; *prove that* $AB \cdot AC = BD \cdot DC - AD^2$.

* 1343. *If from the vert. $\angle$ of a $\triangle$ a st. line be drawn $\perp$ to the base, the rect. contained by the sides of the $\triangle$ shall be equal to the rect. contained by the $\perp$ and the diam. of the circum-$\odot$.* (Euc. VI. C.)

* 1344. *The rect. contained by the diags. of a cyclic quadl. = the sum of the rects. contained by the opp. sides.* (Euc. VI. D.)

* 1345 **PQR** is an equilat. $\triangle$ and **S** is a pt. on the arc **QR** of the circum-$\odot$; prove that $PS = QS + RS$.

* 1346. *If 2 $\triangle$s* **PQR, STV** *have* $\angle Q = \angle T$ *prove that*

$$\triangle PQR : \triangle STV = PQ \cdot QR : ST \cdot TV.$$

* 1347. Two $\odot$s intersect at **P** and **Q** and at **P** tangents are drawn to each $\odot$ meeting the circumfces. at **R** and **S**; prove that $\triangle PQR : \triangle PQS = QR : QS$.

* 1348. **QR** is a diam. of a $\odot$; **P** is a pt. on the circumfce.; the tangents at **P** and **Q** meet at **T**; **PS** is the $\perp$ from **P** to **QR**; prove that **TR** bisects **PS**.

Theorem 57.

(Euc. VI. 6.)

Gen. Enun. *If the ratio of two sides of one triangle be equal to the ratio of two sides of another triangle, and if the angles contained by these sides be equal, the triangles are similar.*

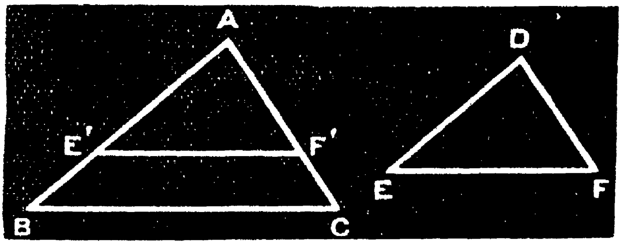


Fig. 356.

Part Enun. Let ABC, DEF be 2 $\triangle$s in which

$$\begin{cases} \angle A = \angle D \\ \dfrac{AB}{AC} = \dfrac{DE}{DF}. \end{cases}$$

It is reqd. to prove that

$\triangle$*s* ABC, DEF *are similar.*

Const. $\because \angle A = \angle D$.

$\triangle$DEF can be placed so that pt. D coincides with pt. A, side DE falls along side AB, and side DF along side AC.

Let E′, F′ be the new positions of E, F.

Proof. $\because \dfrac{AB}{AC} = \dfrac{DE}{DF}$ . . . . . . Hyp.

$\dfrac{AB}{DE} = \dfrac{AC}{DF}$ . . . . Alternando.

that is $\dfrac{AB}{AE'} = \dfrac{AC}{AF'}$ . . . . . . Const.

$\therefore$ E′F′ is $\parallel$ to BC . . . . . Th. 54, Cor.

$\therefore \angle AE'F' = \angle ABC$ and $\angle AF'E' = \angle ACB$ Th. 6.

$\therefore \triangle$s ABC, AE′F′ are equiang.

Hence $\triangle$s ABC, DEF „ „

$\therefore$ each side of the one bears the same ratio to the corresponding side of the other . . . Th. 55.

$\therefore \triangle$s ABC, DEF are similar.

Q.E.D.

Exercises.

1349. If the diags. of a quadl. are divided proportionally at their pt. of intersection, prove that the quadl. is a trapezium.

1350. Two st. lines AB, CD intersect at E so that rect. AE . EB = rect. CE . ED ; prove that △s AED, BEC are similar.

1351. If AD is the ⊥ from the vertex A of a △ ABC to the base BC, and if AB is the mean proportional between BC and BD, prove that ∠A is a rt. ∠.

1352. Prove that the medians from corresponding ∠s of similar △s are equally inclined to the opp. sides.

1353. *From a pt.* D *on the side* AB *of a* △ ABC, DE *is drawn* ∥ *to* BC; *if* $\frac{DE}{BC} = \frac{AD}{AB}$ *prove that the pt.* E *lies on* AC.

* 1354. Two ⊙s intersect at P and Q, and the tangents to the ⊙s at P meet the circumfces. again at R and S; prove that PR : PS = radius of ⊙ PQR : radius of ⊙ PQS.

* 1355. Two ⊙s intersect at P and Q, and a st. line RPS is drawn meeting the circumfces. at R and S; prove that QR : QS = radius of ⊙ PQR : radius of ⊙ PQS.

* 1356. *Find the locus of the mid pts. of st. lines drawn ∥ to the base of a △, and terminated by the sides.*

* 1357. PQR is a △ and S is a pt. on PR, produced if necessary, such that PS is a third proportional to PR and PQ; prove that PQ is a tangent at Q to the circum-⊙ of △ QSR.

* 1358. *From any pt.* P *in a given st. line* OX, PQ *is drawn* ∥ *to another given st. line* OY *so that* PQ *bears a const. ratio to* OP; *find the locus of* Q.

* 1359. The corresponding sides AB, DE of similar △s ABC, DEF are divided at P, R so that BP : AP = ER : DR and the corresponding sides BC, EF at Q, S so that BQ : CQ = ES : FS. Prove that PQ : RS = AC : DF.

* 1360. O is a fixed pt. and PQ a fixed st. line; OR is any st. line drawn from O to PQ, and S is a pt. in OR such that the rect. OR . OS is const. Find the locus of S.

* 1361. A pt. P moves on the arc of a segment of a ⊙ whose chd. is AB. A pt. C is taken on BP such that $\frac{PC}{PA}$ is a const. ratio, and a pt. D is taken on AP such that $\frac{PD}{PB}$ is the same ratio. Prove that CD always touches a fixed ⊙.

THEOREM 58.

(Euc. VI. 3 and A.)

Gen. Enun. *The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle, and likewise the external bisector externally.*

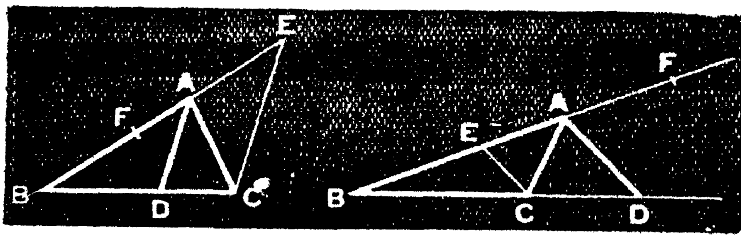


FIG. 357. FIG. 358

Part. Enun. In the △ ABC let AD bisect ∠BAC internally (Fig. 357), and externally (Fig. 358), and cut BC or BC produced at D.

It is reqd. to prove in both cases that

$$\frac{BD}{DC} = \frac{BA}{AC}$$

Const. Through C suppose CE drawn ∥ to DA, cutting BA produced (Fig. 357), and BA (Fig. 358) at E.

Take a pt. F in BA (Fig. 357), and in BA produced (Fig. 358).

Proof. ∵ DA is ∥ to CE . . . . Const.

∴, in both Figs., ∠DAC = ∠ACE and

∠FAD = ∠AEC . . . . Th. 6

But ∠DAC = ∠FAD . . . . Hyp.

∴ ∠ACE = ∠AEC.

∴ AE = AC . . . . . . Th. 13.

Again ∵ DA is ∥ to CE a side of the △BCE . . Const.

∴, in both Figs., $\frac{BD}{DC} = \frac{BA}{AE}$ . . . . Th. 54.

that is $\frac{BD}{DC} = \frac{BA}{AC}$.

Q. E. D.

Def. 71. A straight line AB divided internally at P, and ex-

FIG. 359.

ternally at Q, so that $\frac{AP}{BP} = \frac{AQ}{BQ}$ is said to be divided harmonically.

Exercises.

1362. The base of a $\triangle$ is divided harmonically by the internal and external bisectors of the vert. $\angle$.

1363. AD is a median of a $\triangle$ ABC; DE bisecting $\angle$ ADB cuts AB at E and DF bisecting $\angle$ ADC cuts AC at F. Prove that EF is $\parallel$ to BC.

1364. AD is drawn to the base BC of a $\triangle$ ABC; DE bisecting $\angle$ ADB cuts AB at E and DF bisecting $\angle$ ADC cuts AC at F; if EF is $\parallel$ to BC prove that AD is a median of the $\triangle$ ABC.

1365. *If AB is divided harmonically at P and Q prove that PQ is divided harmonically at A and B.*

1366. If AB is divided harmonically at P and Q prove that

$$\frac{1}{AP} + \frac{1}{AQ} = \frac{2}{AB}.$$

1367. If O is any pt. in a $\triangle$ ABC and OX, OY, OZ bisecting $\angle$s BOC, COA, AOB meet BC, CA, AB in X, Y, Z respy. prove that

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

1368. ABC is a $\triangle$ having AB $>$ AC. If AD bisecting $\angle$ A cuts BC at D and E is the mid. pt. of BC prove that

$$\frac{AB + AC}{AB - AC} = \frac{BC}{ED}.$$

1369. PQR is a $\triangle$ rt. $\angle$d at P and PS, PT are drawn equally inclined to PQ; if PS meets QR in S and RQ produced in T prove that

$$TR : SR = TQ : SQ.$$

1370. If I is the incentre of the $\triangle$ ABC and AI produced cuts BC at D, prove that $\frac{AI}{ID} = \frac{AB + AC}{BC}$.

1371. If the bisectors of the $\angle$s A and C of a quadl. ABCD meet on the diag. BD, prove that the bisectors of the $\angle$s B and D will meet on the diag. AC.

1372. Prove that the internal bisectors of the angles of a $\triangle$ are concurrent.

* 1373. ABCD is a quadl. BD bisects $\angle$ B and BD is the mean proportional between AB and BC. If BD cuts AC at E prove that AE : EC = AD² : CD².

* 1374. If a pt. moves so that its distances from 2 fixed pts. are in a const. ratio, prove that its locus is a ⊙.

* 1375 SRT is a chd. ⊥ to a diam. PRQ of a ⊙; X is a pt. in ST and PX, QX are produced to cut the ⊙ at Z, V. Prove that

$$\frac{VS}{VT} = \frac{ZS}{ZT}.$$

* 1376. BD, the internal bisector of the vert ∠ of a △ ABC, cuts the circum-⊙ at D; prove that $\frac{AB + BC}{BD} = \frac{AC}{AD}$.

* 1377. P is a pt. on a ⊙ whose diam. is QR; prove that the ⊥ from P to QR and the tangent to the ⊙ at P divide QR harmonically.

* 1378. The internal bisector of ∠ A of △ ABC cuts BC at D. If DE ∥ to BA cuts AC at E and DF ∥ to CA cuts AB at F, prove that $BF : CE = AB^2 : AC^2$.

* 1379. Two ⊙s touch internally at P and a chd. QR of the outer ⊙ touches the inner ⊙ at X; if PQ, PR cut the inner ⊙ at S, T prove that $\frac{PS}{PT} = \frac{QX}{RX}$.

THEOREM 59.

(Euc. VI. 19.)

Gen. Enun. *The areas of similar triangles are to one another as the squares on corresponding sides.*

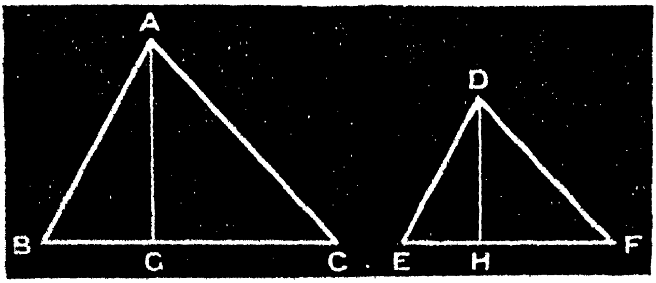


FIG. 360.

Part. Enun. Let ABC, DEF be similar $\triangle$s in which $\angle B = \angle E$ and $\angle C = \angle F$.

It is reqd. to prove that

$$\frac{\triangle ABC}{\triangle DEF} = \frac{BC^2}{EF^2}$$

Const. Suppose AG, DH drawn $\perp$ to BC, EF respy.

Proof. $\because$ $\triangle$s ABG, DEH are equiang. . . . Th. 8, Cor. 2

$\therefore \frac{AG}{DH} = \frac{AB}{DE}$ . . . . Th. 55.

Again $\because$ $\triangle$s ABC, DEF are equiang. . . . . Hyp.

$\therefore \frac{AB}{DE} = \frac{BC}{EF}$ . . . . Th. 55.

Hence $\frac{AG}{DH} = \frac{BC}{EF}$

$\therefore \frac{AG \,.\, BC}{DH \,.\, EF} = \frac{BC^2}{EF^2}$

$$\left.\begin{array}{l}\text{But } AG \,.\, BC = 2 \,.\, \triangle ABC \\ \text{And } DH \,.\, EF = 2 \,.\, \triangle DEF\end{array}\right\} \quad \text{. . . . Th. 28.}$$

$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{BC^2}{EF^2}$

Q.E.D.

Exercises.

1380. In a rt. $\angle$d $\triangle$ if a $\perp$ is drawn from the rt. $\angle$ to the hypot. the $\triangle$s on each side of it are to one another as the sqs. on the sides containing the rt. $\angle$.

1381. E, F are the feet of the $\perp$s from B, C on to the sides AC, AB of a $\triangle$ ABC. Prove that $\triangle ABE : \triangle ACF = AB^2 : AC^2$.

1382. PQ, PR are tangents to a $\odot$ at Q and R. If S is the centre of the $\odot$ prove that $\triangle PQR : \triangle QSR = PQ^2 : SQ^2$.

1383. PQ, RS are chds. of a $\odot$ that intersect when produced at a pt. T outside the $\odot$. Prove that $\triangle PRT : \triangle QST = PR^2 : QS^2$.

1384. The side AB of a $\triangle$ ABC is divided at P so that $\dfrac{AP}{AB} = \dfrac{1}{\sqrt{2}}$. Prove that a line through P $\parallel$ to BC will bisect the $\triangle$ ABC.

1385. D, E are the mid. pts of the sides AB, AC of a $\triangle$ ABC and CD, BE intersect at F. Prove that $\triangle BFC = 4 \triangle DFE$.

1386. The areas of similar $\triangle$s are to one another as the sqs. on their circum-radii.

* 1387. *In any rt. $\angle$d $\triangle$ the area of a $\triangle$ descd. on the hypot. = the sum of the areas of similar $\triangle$s descd. on the sides containing the rt. $\angle$.*

* 1388. PT is the tangent at P to a $\odot$ whose centre is O; a diam. QOR when produced meets PT at T; the tangent at R meets PT at S; prove that $\triangle RST : \triangle PTO = TR : TQ$.

* 1389. D, E are the feet of the $\perp$s from A, C to the sides BC, AB of a $\triangle$ ABC. Prove that $\triangle ABC : \triangle DBE = AB^2 : BD^2$.

* 1390. If the areas of 2 isos. $\triangle$s are to one another as the sqs. on their bases, prove that they are similar.

* 1391. PSR is a diam. of a $\odot$ whose centre is O, and S is the foot of the $\perp$ upon it from any pt. Q on the circumfce. If the tangents at P and Q meet at T, prove that $\triangle PTQ : \triangle QOR = PS : SR$.

* 1392. ABC is a $\triangle$ rt. $\angle$d at A and BCP, CAQ, ABR are equilat. $\triangle$s descd. on BC, CA, AB respy. If D is the foot of the $\perp$ from A on BC, prove that $\triangle ABR = \triangle BPD$ and $\triangle ACQ = \triangle CPD$.

PRACTICAL SECTION.

PROPORTIONALS.

PROBLEM 25.

(Euc. VI. 12.)

Gen. Enun. *To find the fourth proportional to three given straight lines.*

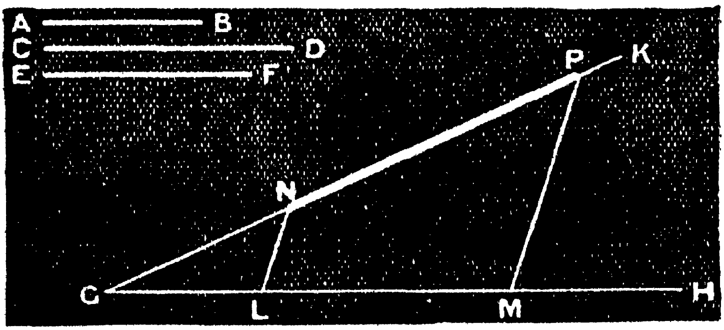


FIG. 86.

Part. Enun. Let AB, CD, EF be 3 given st. lines.
It is reqd. to find the fourth proportional to AB, CD, EF.

Const. Draw 2 st. lines GH, GK making any convenient $\angle$.
From GH cut off GL = AB and LM = CD.
From GK cut off GN = EF.
Join LN.
Through M draw MP $\parallel$ to LN cutting GK at P . . . Prob. 6.
Then shall NP be the fourth proportional to AB, CD, EF.

Proof. $\because$ LN is $\parallel$ to MP a side of $\triangle$ GPM . . . Const.

$\therefore \dfrac{GL}{LM} = \dfrac{GN}{NP}$ . . . . Th. 54.

But GL = AB, LM = CD, GN = EF.

$\therefore \dfrac{AB}{CD} = \dfrac{EF}{NP}$.

That is NP is the fourth proportional to AB, CD, EF.

Q.E.F.

Cor. *To find the third proportional to two given straight lines* (Euc. VI. 11.)

Exercises.

1393. Find by construction the 3rd proportional to 3·7 in. and 2·8 in and verify by measurement and calculation.

1394 Find by construction the 4th proportional to 4·3 cm., 6·6 cm. 7·1 cm., and verify by measurement and calculation.

1395. Find by construction the following lengths and verify by measurement and calculation :—

(1) $\dfrac{3 \cdot 9 \times 2 \cdot 5}{4 \cdot 7}$ in.

(2) $\dfrac{2 \cdot 3 \times 5 \cdot 1}{6 \cdot 7}$ cm.

(3) $1 \cdot 1 \times 4 \cdot 3$ in.

(4) $\dfrac{1}{0 \cdot 9}$ in.

1396. Divide a given st. line internally into segments proportional to 2 given st. lines.

1397. Divide a given st. line externally into segments proportional to 2 given st. lines.

1398. Divide a given st. line into segments proportional to 3 given st. lines.

1399. Divide a given st. line into segments proportional to the segments of a given divided st. line. (Euc. VI. 10.)

* 1400. Given a st. line representing a in., draw lines to represent the following lengths :—

(1) a^2 in.
(2) a^3 in.
(3) a^4 in.

* 1401. Construct a △ of given perimeter whose sides are proportional to 3 given st. lines. When is the construction impossible?

* 1402. AB, AC are st. lines of indefinite length and P is a pt. between them. Through P draw a st. line terminated by AB at Q and by AC at R so that $\dfrac{PQ}{PR} = \frac{2}{3}$.

* 1403. Construct a △ of given base, of one given base ∟ and whose sides are proportional to 2 given st. lines. Show that there may be 2 such △s. When is there only one, and when is the construction impossible?

* 1404. Construct a △ of given base, of given vert. ∟ and whose sides are proportional to 2 given st. lines.

PROBLEM 26.

(EUC. VI. 13.)

Gen. Enun. *To find the mean proportional between two given straight lines.*

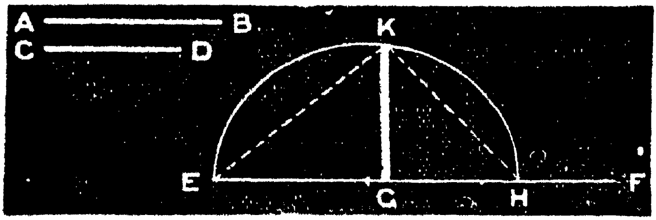


FIG. 362.

Part. Enun. Let AB, CD be 2 given st. lines.
It is reqd. to find the mean proportional between AB, CD.

Const. From any st. line EF of indefinite length cut off EG = AB, GH = CD.
On EH as diam. desc. a semi-⊙ EKH.
From G draw GK ⊥ to EH meeting the semi-⊙ EKH at K . . . . Prob. 3.
Then shall GK be the mean proportional between AB, CD.

Proof. Join EK, KH.
∵ EKH is a semi-⊙ . . . . . . Const.
∴ $\angle$EKH is a rt. $\angle$ . . . . . . Th. 45.
∴ $\angle$EKG = 90° − $\angle$HKG
= $\angle$KHG.
∴ △s EKG, KHG are equiang. . . Th. 8, Cor. 2.
∴ $\frac{EG}{GK} = \frac{GK}{GH}$ . . . . . . Th. 55.
But EG = AB, GH = CD.
∴ $\frac{AB}{GK} = \frac{GK}{CD}$.
That is GK is the mean proportional between AB, CD.

Q. E. F.

Exercises.

1405. Find by construction the mean proportional between 37 mm and 43 mm., and verify by measurement and calculation.

1406. Represent graphically the following lengths and verify by calculation and measurement to 2 places of decimals :—

(1) $\sqrt{1{\cdot}3 \times 1{\cdot}7}$ in.
(2) $\sqrt{8}$ cm.
(3) $\sqrt{5{\cdot}1}$ in.
(4) $2\sqrt{3}$ cm.
(5) $\sqrt[4]{11}$ cm.
(6) $(\sqrt{5})^5$ mm.

MAXIMA AND MINIMA.

If a geometrical magnitude varies its position continuously according to any law its value may or may not vary with its position.

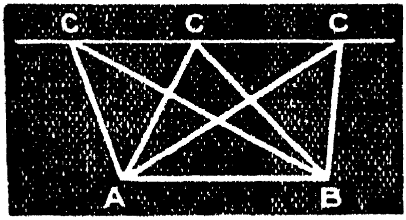


FIG. 363.

In Fig. 363, for example, the position of the triangle ABC varies continuously if its vertex C is supposed to move along a straight line parallel to its fixed base AB. But the area of the triangle does not vary with its position, and so is said to be constant.

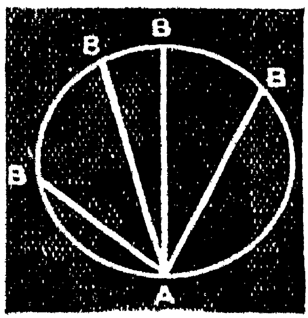


FIG. 364.

In Fig. 364 the position of the chord AB varies continuously if its extremity B is supposed to move along the circumference of the circle, its other extremity A remaining fixed. In this case, however, the length of the chord varies with its position and increases until it becomes a diameter of the circle when it begins to decrease.

In Fig. 365 the position of the line AB varies continuously if its extremity B is supposed to move along the straight line XY, its other extremity A remaining fixed. In this case, also, the length of the line varies with its position and decreases until it becomes the perpendicular to XY, when it begins to increase.

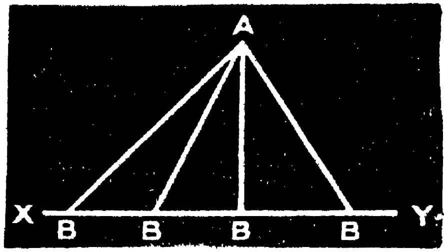


FIG. 365.

Def. 72. When a geometrical magnitude, varying its position continuously according to any law, stops increasing and begins to decrease its value is said to be a maximum, and when it stops decreasing and begins to increase its value is said to be a minimum.

Hence it follows that a maximum value of a continuously moving geometrical magnitude is greater while a minimum value is less than the values which immediately precede and follow.

Exercises.

1407. *If* A, B *are two fixed points on the same side of a fixed straight line* XY *and* P *is a moving point on* XY, *then* AP + PB *is a minimum when* P *occupies a position such that* $\angle$APX = $\angle$BPY.

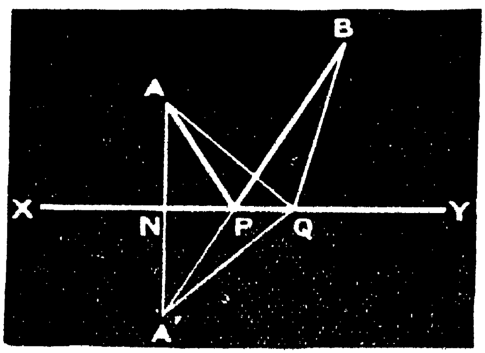


FIG. 366.

Const. and Proof. Suppose $\angle$ APX = $\angle$ BPY and ANA′ drawn $\perp$ to XY meeting BP produced at A′.

Join Q, which is any pt. in XY other than P, to A, B, A′.

Now $\because$ $\triangle$ APN $\equiv$ $\triangle$ A′PN . . (Why?)

$\therefore$ AP + PB = A′P + PB = A′B.

Again $\because \triangle AQN \equiv \triangle A'QN$ . . (Why?)
$\therefore AQ + QB = A'Q + QB$.
But $A'B < A'Q + QB$ . . . Th. 18.
$\therefore AP + PB < AQ + QB$.

Q. E. D.

Note. Fig. 366 illustrates an important law in Optics, namely that a ray of light AP incident upon a plane mirror XY will be reflected along PB. Hence in passing from A to B by reflection in the mirror the light travels by the shortest possible path.

1408. Given the base of a $\triangle$ and its area find the position of its vertex so that the sum of its sides may be a minimum.

1409. Given the area of a 5-sided polygon prove that the perimeter is least when it is equilat.

1410. Given the base of a $\triangle$ and the sum of its sides, prove that its area is a maximum when it is isosceles.

1411. Given the perimeter of a $\triangle$ prove that its area is a maximum when it is equilat.

1412. *If* A, B *are two fixed points on opposite sides of a fixed straight line* XY *and* P *is a moving point on* XY, *then the difference between* AP *and* PB *is a maximum when* P *occupies a position such that* $\angle APX = \angle BPX$

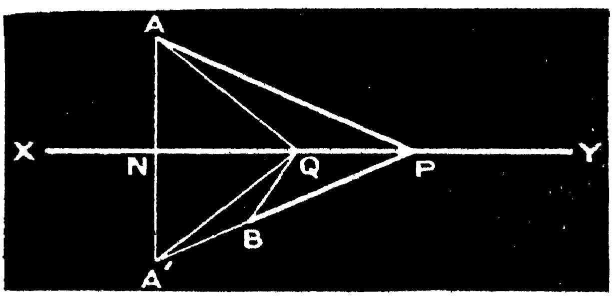


FIG. 367.

Const. and Proof. Suppose $\angle APX = \angle BPX$ and ANA′ drawn $\perp$ to XY meeting PB or PB produced at A′.

Join Q which is any pt. in XY other than P, to A, B, A′.

Now $\because \triangle APN \equiv \triangle A'PN$ (Why?)

$\therefore$ diffce. between AP and PB = diffce. between A′P and PB = A′B.

13

14

Again ∵ △ AQN ≡ △ A′QN (Why?)
∴ diffce between AQ and QB = diffce. between A′Q and QB.
But A′B > diffce. between A′Q and QB . . . . Th. 18, Cor.
∴ diffce. between AP and PB > diffce. between AQ and QB.

Q.E.D.

1413. *If AB is a finite straight line and P a moving point on a fixed straight line XY of indefinite length, ∠ APB will be a maximum when ⊙ APB touches line XY.*

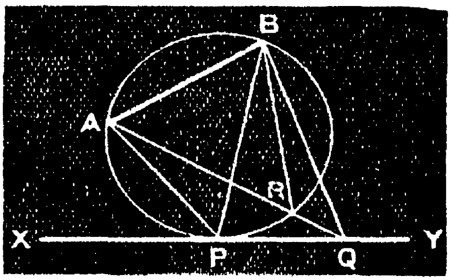


FIG. 368.

Const. and Proof. Suppose ⊙ APB touches line XY. Take Q which is any pt. on XY other than P and on the same side of AB as P.
Join QA, QB.
Let QA cut ⊙ APB at R.
Join BR.
Now ∠ APB = ∠ ARB . . Th. 43.
But ∠ ARB > ∠ AQB . Th. 8, Cor. 4.
∴ ∠ APB > ∠ AQB.

Q.E.D.

Note.—Since two circles can be described to pass through A, B and to touch XY there are two points in XY at which the angle subtended by AB will be a maximum but they are on opposite sides of AB.

1414. AB is a finite st. line outside a given ⊙. Find a pt. on the circumfce. of the ⊙ at which AB subtends, (1) the maximum angle, (2) the minimum angle.

1415. *If* P *is any pt. in a given st. line* AB, *rect.* AP . PB *is a maximum when* P *is the mid. pt. of* AB.

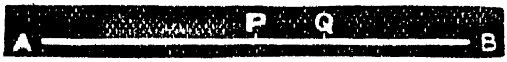


FIG. 369.

Proof. If Q is any pt. in AB other than the mid. pt. P

$AQ \cdot QB + PQ^2 = AP^2$ . . Th. D, Cor. 1 (p. 326).

$= AP \cdot PB$

$\therefore AP \cdot PB > AQ \cdot QB$. Q.E.D.

1416. **Of all rectangles of given perimeter the square has the maximum area.**

1417. *If* P *is any point in a given straight line* AB, $AP^2 + BP^2$ *is a minimum when* P *is the middle point of* AB.

FIG. 370.

Proof. If Q is any pt. in AB other than the middle pt. P

$AQ^2 + BQ^2 = 2AP^2 + 2PQ^2$. Th. F, Cor. 1 (p. 331).

$= AP^2 + BP^2 + 2PQ^2$

$\therefore AP^2 + BP^2 < AQ^2 + BQ^2$. Q.E.D.

1418. *Of all rectangles of given area, the square has the minimum perimeter.*

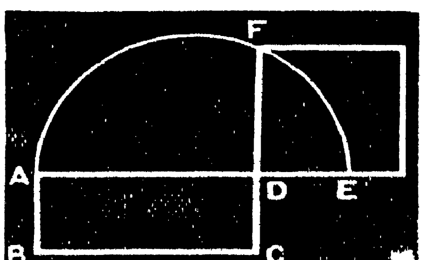


FIG. 371.

Const. and Proof. Let ABCD be a rect. of the given area.
Construct the equivalent sq. DF^2 . Prob. 24.
Now AE, the diam. of the semi-⊙ AFE, is half the perimeter of the rect. ABCD.

And 2DF is half the perimeter of the sq. DF^2.
But of all st. lines that can be drawn in a ⊙ the diam. is the greatest.
$\therefore$ AE > 2DF.
And 2AE > 4DF.

Q.E.D.

It follows from Def. 72 that if a geometrical magnitude, varying its position continuously, can occupy two positions of equal value, there must be an intermediate position of maximum or minimum value which can be determined by making the magnitudes of equal value gradually approach (while always remaining equal) and ultimately coincide.

Many of the Exercises that follow are solved by the application of this principle.

Exercises.

1419. *Find the minimum straight line from a point* P *to a straight line* AB *of indefinite length.*

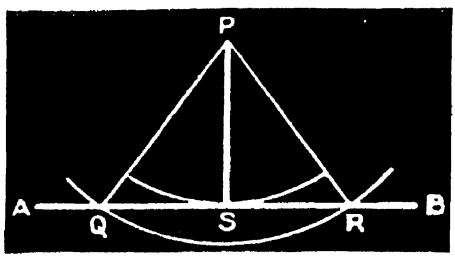


FIG. 372.

Const. and Proof. A ⊙ descd. with centre P so as to cut AB will meet it in two pts. Q and R such that PQ = PR.
$\therefore$ by diminishing the radius of the ⊙ so that the equal lengths PQ and PR gradually approach (while always remaining equal) and ultimately coincide in PS, we arrive at a max. or min. st. line from P to AB.
This happens when the ⊙ touches AB and $\therefore$ PS is ⊥ to AB . Th. 40
The Fig. shows that PS cannot be a maximum $\therefore$ it must be a minimum.

J.E.F.

1420. Find (1) the maximum (2) the minimum st. line from a given pt. to the circumfce. of a given ⊙.

1421. *Through a point P within a given circle of centre Q find the minimum chord of the circle.*

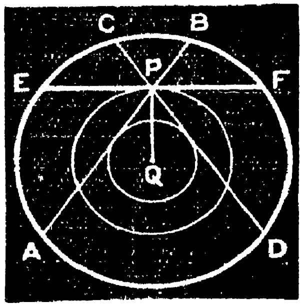


FIG. 373.

Const. and Proof.	Desc. a ⊙ with centre Q and radius $<$ PQ.	
	Through P draw AB and CD tangents to this ⊙ and chds. of the given ⊙	Prob. 16.
	Then AB = CD	Th. 39.
	∴ by increasing the radius of the inner ⊙ so that the = chds. AB, CD gradually approach (while always remaining equal) and ultimately coincide in EF, we arrive at a max. or min. chd. through P.	
	This happens when the inner ⊙ passes through P and ∴ EF is ⊥ to PQ	Th. 40.
	The Fig. shows that EF cannot be a maximum ∴ it must be a minimum.	

Q.E.F.

Notice that by decreasing the radius of the inner ⊙ the equal chords AB and CD can be made to gradually approach (while always remaining equal) and ultimately coincide in a diameter of the given ⊙ which is evidently the maximum chord through P.

1422. *Find the triangle of maximum area having a given base* AB *and a given vertical angle* α.

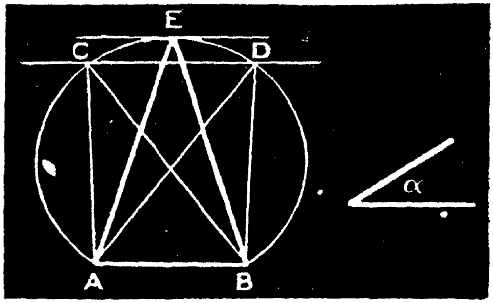


FIG. 374.

Const. and Proof. On AB desc. a segment of a ⊙ containing an ∟ = ∟a . . . . Prob. 21.

Draw any st. line ∥ to AB cutting the arc of this segment at C and D.

Join AC, BC, AD, BD.

Now △s ABC, ABD have each the given base and the given vert. ∟.

And are equal in area . Th. 28, Cor. 2.

∴ by increasing the distance between the parallels CD and AB so that the equivalent △s ABC, ABD gradually approach (while always remaining equal) and ultimately coincide in △ AEB, we arrive at a max. or min. △ on base AB and having vertical ∟ = ∟a.

This happens when CD touches the ⊙ at E and it can easily be shown that △ AEB is isosceles.

The Fig. shows that area of △ AEB cannot be a minimum

∴ it must be a maximum. Q.E.F.

1423. A ladder leans against a vertical wall; in what position does it intercept with the vertical wall and the horizontal floor a △ of max. area?

1424. Of all rectangles that can be inscribed in a ⊙ the square has the max. area.

1425. Find the △ of max. vert. ∟ having a given base and a given area.

1426. Inscribe a rect. of max. area in a given semi-⊙.

1427. *Construct the triangle of maximum area, its two sides* AB *and* AC *being given.*

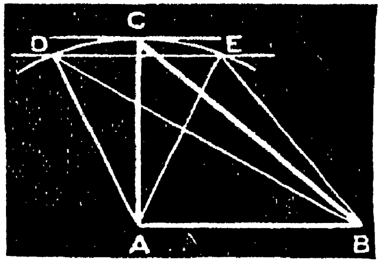


FIG. 375.

Const. and Proof. Desc. a ⊙ with centre A and radius = AC and suppose it to meet a parallel to AB at D and E.

Join AD, BD, AE, BE.

Now △s ABD, ABE have each two sides equal to AB, AC respy.

And are equal in area . Th. 28, Cor. 2.

∴ by increasing the distance between the parallels DE and AB so that the equivalent △s ABD, ABE gradually approach (while always remaining equal) and ultimately coincide in △ ACB we arrive at a max. or min. △ having sides of the given lengths.

This happens when DE touches the ⊙ at C, and it can easily be shown that AC is ⊥ to AB.

The Fig. shows that area of △ ACB cannot be a minimum.

∴ it must be a maximum. Q.E.F.

1428. Construct the parallelogram of maximum area, its two adjacent sides AB and AC being given.

* 1429. From a pt. A outside a ⊙ of centre K draw a st. line to cut the circumfce. in B and C so that the △ BCK may be of max. area.

* 1430. Construct the ▱gm. of max. area its 2 diags. being given.

* 1431. Through a pt. of intersection of 2 ⊙s draw the max. st. line terminated by the circumfces.

* 1432. Find the pt. in a given st. line of indefinite length from which the tangents drawn to a given ⊙ contain the max. ∟.

* 1433. Through a given pt. P between 2 intersecting st. lines OX, OY draw a line which shall form with OX and OY a △ of min. area

* 1434. Through a given pt. P between 2 intersecting st. lines OX, OY draw a line meeting OX and OY in A and B respy. so that rect. AP . BP may be a minimum.

* 1435. Construct the △ of min. area having given its altitude and vert. ∟.

* 1436. AB is a finite st. line outside a given ⊙. Find a pt. P on the circumfce. of the ⊙ such that area of △ APB may be (1) a maximum, (2) a minimum.

* 1437. If AB is a chd. of a given ⊙, and C is any pt. on the circumfce., find the position of C so that the perimeter of △ ABC may be a maximum.

* 1438. Of all △s that can be inscribed in a ⊙ the equilateral has the maximum perimeter.

DEFINITIONS, POSTULATES, AXIOMS AND HYPOTHETICAL CONSTRUCTIONS.

Definitions.

1. A *point* is that which has position but no length, breadth or thickness. (P. 32.)

2. A *line* has position and length but no breadth or thickness. (P. 32.)

3. A *straight line* lies evenly between its extreme points. (P. 33.)

4. A *surface* has position, length and breadth but no thickness. (P. 33.)

5. A *plane surface* or *plane* is a surface in which, any two points being taken, the straight line that joins them lies wholly in that surface. (P. 33.)

6. A *solid* has position, length, breadth and thickness. (P. 33.)

7. When two straight lines meet in a point they are said to make with each other a *plane rectilineal angle* or, shortly, an *angle*. (P. 34.)

8. Two angles are said to be *adjacent* when they have a common vertex and lie on opposite sides of a common arm. (P. 35.)

9. A *plane figure* is a part of a plane bounded by one or more lines. (P. 35.)

10. When a straight line standing on another straight line makes the adjacent angles equal to one another each of these angles is called a *right angle*, and the straight line which stands on the other is called a *perpendicular* to it. (P. 37.)

11. An *obtuse angle* is greater than one right angle but less than two right angles. (P. 37.)

12. An *acute angle* is less than a right angle. (P. 37.)

13. A *circle* is a plane figure contained by one line which is called the *circumference* and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another. This point is called the *centre* of the circle. (P. 38.)

14. Any straight line drawn from the centre of a circle to its circumference is called a *radius* of the circle, and any straight line drawn through the centre of a circle and terminated both ways by the circumference is called a *diameter* of the circle. (P. 38.)

15. The *bisector* of a magnitude is that which divides it into two equal parts—the *trisectors* into three equal parts. (P. 38.)

16. When two angles are together equal to two right angles each is said to be the *supplement* of the other and the two angles are said to be *supplementary*. (P. 47.)

17. When two angles are together equal to one right angle, each is said to be the *complement* of the other and the two angles are said to be *complementary*. (P. 47.)

18. A *corollary* is a geometrical truth that can be easily deduced from a proved proposition. (P. 47.)

19. Two theorems are said to be *converse* each of the other when the hypothesis of each is the conclusion of the other. (P. 49.)

20. Of the four angles formed by two intersecting straight lines those that are opposite one another are called *vertically opposite angles*. (P. 50.)

21. Straight lines in the same plane, which do not meet, however far they are produced in either direction, are said to be *parallel*. (P. 52.)

22. A *triangle* is a figure bounded by three straight lines. (P. 60.)

23. An *equilateral triangle* is one which has its three sides equal. (P. 60.)

24. An *isosceles triangle* is one which has two sides equal. (P. 60.)

25. A *scalene triangle* is one which has three unequal sides. (P. 61.)

26. A *right-angled triangle* is one which has a right angle. (P. 61.)

27. An *obtuse-angled triangle* is one which has an obtuse angle. (P. 61.)

28. An *acute-angled triangle* is one which has three acute angles. (P. 61.)

29. A *quadrilateral* is a plane figure bounded by four straight lines. (P. 62.)

30. A *polygon* is a plane figure bounded by four or more straight lines. (P. 62.)

31. A *convex polygon* has each of its angles less than two right angles. (P. 62.)

32. A *regular polygon* has all its sides equal and all its angles equal. (P. 62.)

33. The *diagonals of a polygon* are straight lines joining opposite corners. (P. 63.)

34. Figures which can be made by superposition to coincide or fit exactly are said to be *congruent*. (P. 69.)

35. Three or more lines are said to be *concurrent* when they meet in the same point. (P. 69.)

36. Three or more points are said to be *collinear* when they lie on the same straight line. (P. 69.)

37. An *axis of symmetry* of a figure is a line about which the figure can be folded so that one half may coincide with the other half. (P. 75.)

38. The *distance* of a point from a straight line is the perpendicular from the point on to the line. (P. 99.)

39. A *parallelogram* is a quadrilateral whose opposite sides are parallel. (P. 101.)

40. A *rhombus* is a parallelogram having two adjacent sides equal. (P. 101.)

41. A *rectangle* is a parallelogram having one of its angles a right angle. (P. 101.)

42. A *square* is a rectangle having two adjacent sides equal. (P. 102.)

43. A *trapezium* is a quadrilateral having only one pair of opposite sides parallel. (P. 102.)

44. An *isosceles trapezium* is a trapezium in which the sides that are not parallel are equal to one another. (P. 102.)

45. The *orthogonal projection* of one straight line on another of unlimited length is the portion of the latter intercepted between the perpendiculars drawn to it from the extremities of the former. (P. 102.)

46. If every point on a line, or group of lines, satisfies a given condition, and no other point does so, then that line or group of lines is called the *locus* of the point satisfying that condition. (P. 111.)

47. The parallels to the sides of a parallelogram through any point in one of the diagonals divide it into four parallelograms, of which the two through which the diagonal passes are called the *parallelograms about a diagonal*, and the other two are called the *complements of the parallelograms about a diagonal.* (P. 222.)

48. A *chord* of a circle is a straight line joining any two points on the circumference. (P. 231.)

49. An *arc* of a circle is any part of the circumference. (P. 231.)

50. A *sector* of a circle is a figure bounded by two radii and the arc intercepted between them. (P. 231.)

51. A *figure* is said to be *symmetrical about a line* if the line divides it into two parts which coincide when the figure is folded about the line. The line is called the *axis of symmetry* of the figure. (P. 232.)

52. A *figure* is said to be *symmetrical about a point* if the point bisects every line drawn through it to meet the boundary of the figure in both directions. The point is called the *centre of symmetry* of the figure. (P. 232.)

53. A *circle* is said to be *circumscribed* about a rectilineal figure, and the *figure* is said to be *inscribed* in the circle if the circle passes through all the vertices of the rectilineal figure. (P. 237.)

54. *Points* are said to be *concyclic* when they lie on the same circle. (P. 237.)

55. A *secant* of a circle is a straight line of indefinite length which meets the circumference of the circle in two points. (P. 247.)

56. A *tangent* to a circle is a straight line which meets the circumference of the circle in one, and only one, point, however far produced. This straight line is said to *touch* the circle, and the point where it meets the circle is called the *point of contact.* (P. 247.)

57. A *tangent* to a circle is a straight line which meets the circumference of the circle in two consecutive points. (P. 248.)

58. Two *circles* which meet in one, and only one, point, are said to *touch* one another, and the point at which they meet is called the *point of contact.* (P. 249.)

59. Two *circles* which meet in two consecutive points are said to *touch* one another. (P. 249.)

60. *Common tangents* to two circles are said to be *direct* or *transverse* according as the two circles lie on the same side or on opposite sides of the common tangent. (P. 250.)

61. Two *circles* are said to *cut orthogonally* when the two tangents at either point of intersection are at right angles to one another. (P. 251.)

62. A *circle* is said to be *inscribed* in a rectilineal figure, and the *figure* is said to be *circumscribed* about the circle if the circle touches all the sides of the rectilineal figure. (P. 251.)

63. A *segment* of a circle is the figure bounded by a chord of the circle and one of the arcs into which it divides the circumference. (P. 261.)

64. An *angle in a segment* is the angle subtended by the chord of the segment at a point on the arc. (P. 262.)

65. A *cyclic quadrilateral* is one whose vertices lie on the same circle. P. 269.)

66. If a point B is taken in a straight line AC, then AC is said to be *divided internally* at B into the segments AB and BC. If a point B is taken in a straight line AC produced, then AC is said to be *divided externally* at B into the segments AB and BC. (P. 324.)

67. If a straight line AB is divided internally or externally at E so that $AB \cdot BE = AE^2$, then the straight line AB is said to be divided at E in *medial section*. (P. 328.)

68. If the ratio of one magnitude to a second is equal to the ratio of a third magnitude to a fourth, the four magnitudes are said to be *in proportion*, and the first and second are said to be *proportional* to the third and fourth. (P. 353.)

69. If the ratio of one magnitude to a second is equal to the ratio of the second to a third, the three magnitudes are said to be *in proportion*. (P. 353.)

70. If $\frac{A}{B} = \frac{B}{C} = \frac{C}{D} =$ etc., the magnitudes A, B, C, D, etc., are said to be in *continued proportion*. (P. 353.)

71. A straight line AB divided internally at P and externally at Q so that $\frac{AP}{BP} = \frac{AQ}{BQ}$ is said to be *divided harmonically*. (P. 369.)

72. When a geometrical magnitude, varying its position continuously according to any law, stops increasing and begins to decrease its value is said to be a *maximum*, and when it stops decreasing and begins to increase its value is said to be a *minimum*. (P. 378.)

Postulates. (P. 39.)

Let it be granted:—

1. That a straight line may be drawn from any one point to any other point.

2. That a finite straight line may be produced to any length in that straight line.

3. That a circle may be described with any centre and with a radius equal to any finite straight line.

Axioms. (P. 41.)

1. Things which are equal to the same thing are equal to one another.
2. If equals be added to equals the wholes are equal.
3. If equals be taken from equals the remainders are equal.
4. If equals be added to unequals the wholes are unequal.
5. If equals be taken from unequals the remainders are unequal.
6. Things which are halves of the same thing are equal to one another
7. The whole is equal to the sum of its parts.
8. Any magnitude can be transferred from one position to another without its shape or size undergoing any change.
9. Magnitudes which coincide with one another, that is which exactly fill up the same space, are equal to one another.
10. Two straight lines cannot enclose a space.
11. All right angles are equal.
12. (Playfair's Axiom).—Two straight lines that intersect one another cannot both be parallel to the same straight line.

Hypothetical Constructions. (P. 45.)

(*a*) A line or angle can be divided into any number of equal parts.

(*b*) A line can be drawn from any point in any desired direction and of any desired length.

(*c*) A figure can be reproduced or placed in any position.

ANSWERS.

Part II.

Exp. 127. (11, 6), (– 11, 6), (– 11, – 6), (11, – 6), (4, 0), (0, 4), (– 4, 0), (0, – 4), (0, 0).

Exp. 138. (i) 1·1 in., (ii) 0·8 in., (iii) 1·5 in., (iv) 0·9 in., (v) 0·55 in., (vi) 1·21 in.

Exp. 139. (i) (0, 0), (ii) (0, 9), (iii) (– 3, 0), (iv) (16, 0), (v) (0, 8·9), (vi) (1·7, 4·1).

Exp. 141. (i) (5, 5), (ii) (5, 8), (iii) (5·5, 8·5), (iv) (– 2, – 5)

Exp. 142. (2, – 1).

Exp. 145. (6·5, 3·5).

Exp. 147. (3 1, 1) and (7·9, 7).

Exp. 148. (11·8, 6) and (– 1·8, 6).

Exp. 149. (– 2·5, – 1·9).

Exp. 150. (8·7, 0·3).

Exp. 156. (3, 2).

Exp. 157. $x = 0, y = 0$; $x = 2, y = -2$.

Exp. 158. 20 miles an hour, $32\frac{1}{2}$ miles an hour, 35 miles an hour, 20 miles an hour, $35\frac{1}{4}$ miles an hour, $3\frac{3}{4}$ miles from B.

Exp. 163. 100, $\frac{1}{100}$.

Exp. 167. 1·6 sq. in.

Exp. 168. (i) 1·54 sq. in., (ii) 2·21 sq. in., (iii) 1·066 sq. in., (iv) 0·4368 sq. in.

Exp. 169. (i) 3·57 sq. in., (ii) 3·8775 sq. in., (iii) 1·7424 sq. in., (iv) 2·1424 sq. in.

Exp. 170. (i) 3 sq. ft. 33 sq. in., (ii) 26·22 sq. cm., (iii) 83 sq. ft., 48 sq. in., (iv) 3446 sq. yds. 6 sq. ft., (v) xy sq. in., (vi) $3ab$ sq. ft.

Exp. 171. (i) 2 sq. ft. 73 sq. in., (ii) 8 sq. yds. 4 sq. ft. 81 sq. in., (iii) $9x^2$ sq. in., (iv) $(x^2 + y^2 + 2xy)$ sq. cm.

Exp. 172. (i) 5 ft. 8 in., (ii) 121 yds., (iii) $\frac{x}{y}$ in., (iv) $\frac{144p}{q}$ in.

Exp. 176. (i) 13 in., (ii) $\sqrt{x}$ in., (iii) $(a + b)$ miles.

Exp. 180. £3 11s. $1\frac{1}{2}$d.

Exp. 181. £117 6s. 8d.

Exp. 182. 9 yds. 2 in., 6 yds. 1 ft. 7 in.

Exp. 183. 450.

Exp. 185. (i) 0·385 sq. in., (ii) 0·32 sq. in., (iii) 1·0452 sq. in., (iv) 1·10285 sq. in.

Exp. 191. (i) 0·84 sq. in., (ii) 0·77 sq. in., (iii) 1·3 sq. in., (iv) 0·36 sq. in.

Exp. 192. (i) 1·12 sq. in., (ii) 1·1655 sq. in., (iii) 2·2311 sq. in., (iv) 2·5833 sq. in.

Exp. 193. (i) 10 sq. ft., (ii) 17 sq. yds. 5 sq. ft. 126 sq. in., (iii) 1332½ acres, (iv) $12xy$ sq. in.

Exp. 194. (i) 15 chains, (ii) $\frac{9a}{b}$ ft.

Exp. 196. (i) 1·67 sq. in., (ii) 4·06 sq. cm., (iii) 0·84 sq. in., (iv) 3·04 sq. cm.

Exp. 204. (i) 0·6 sq. in., (ii) 0·525 sq. in., (iii) 0·405 sq. in., (iv) 0·385 sq. in.

Exp. 205. (i) 0·675 sq. in., (ii) 0·935 sq. in., (iii) 0·87125 sq. in., (iv) 0·77405 sq. in.

Exp. 206. (i) 1 sq. ft. 135 sq. in., (ii) 540 sq. yds., (iii) 110 acres 1 rood, (iv) $6pq$ sq. in.

Exp. 207. (i) 9 chains 30 links, (ii) 11 yds. 2 ft. 8 in.

Exp. 208. (i) 13 chains 72 links, (ii) $18\sqrt{2}$ ft.

Exp. 210. (i) 16·17 sq. cm., (ii) 1·16 sq. in.

Exp. 216. (i) 9·06 sq. cm., (ii) 1·05 sq. in., (iii) 1·64 sq. in., (iv) 1·9 sq. in., (v) 10·84 sq. cm.

Exp. 221. 0·67 sq. in.

Exp. 222. 0·515 sq. in., 0·675 sq. in.

Exp. 223. (i) 0·455 sq. in., (ii) 0·80 sq. in., (iii) 0·955 sq. in., (iv) 0·605 sq. in.

Exp. 229. (i) 1·48 in., (ii) 1·25 in., (iii) 1·43 in., (iv) 1·66 in.

Exp. 230. (i) 1·43 in., (ii) 1·92 in., (iii) 1·86 in., (iv) 2·15 in.

Exp. 231. Scalene.

Exp. 234. (2, 4).

Exp. 235. 5·92 cm.

Exp. 236. 88 mm.

Exp. 239. 1·41 in.

Exp. 241. 3·54 cm.

Exp. 243. 1·52 in.

Exp. 244. 1·2369 . . . in., 1·298 . . . sq. in.

Exp. 246. 2·14146 . . . cm., 3·139 . . . cm., 6·4 . . . sq. cm.

Exp. 249. (i) 306 sq. yds., (ii) 90 sq. miles, (iii) 16 sq. ft. 6 sq. in. (iv) 64 sq. yds. 8 sq. ft. 12 sq. in.

Exp. 251. 44 mm.

Exp. 258. 0·6 in.

Exp. 259. 0·49 in.

Exp. 260. 5 cm.

Exp. 262. 2·41 in.; 0·97 in.

Exp. 263. 0·49 cm.; 2·37 cm.
Exp. 264. 2·4 in.
Exp. 270. 4·13 cm.
Exp. 272. (i) 0·82 in., (ii) 1·08 in., (iii) 0·48 in., (iv) 0·92 in.
Exp. 278. $\frac{1}{8}$.
Exp. 280. 25·5 mm.
Exp. 281. 1·73 in.
Exp. 283. (i) 154 in., (ii) 88 miles, (iii) 352 mm.
Exp. 284. (i) 14 ft., (ii) 3·5 metres, (iii) 490 miles.
Exp. 285. 59·690 in.
Exp. 286. 47·746 mm.
Exp. 287. 360.
Exp. 288. 104 days 18 hours 17$\frac{1}{7}$ minutes.
Exp. 292. 20 sq. yds. 106 sq. in.
Exp. 293. 39·2 yds.
Exp. 294. 29 acres 4·625 poles.
Exp. 296. 2·89 . . . in.
Exp. 297. £19 0s. 3$\frac{3}{7}$d.
Exp. 298. £75 19s. 5·544d.
Exp. 299. 21·46 sq. in.
Exp. 301. 17·3 mm.
Exp. 302. 0 in., 0·74 in., 0·58 in.
Exp. 304. 1·30 cm.
Exp. 305. 4·8 cm.
Exp. 306. $\frac{5}{11}$.
Exp. 317. 60°.
Exp. 318. 11°, 40°.
Exp. 319. 30°.
Exp. 320. 140°.
Exp. 327. 57°.
Exp. 328. 0·65 in.
Exp. 330. 111°.
Exp. 335. 78°.

Ex. 571. 1·59 in., 2·06 in., 0·93 in.
Ex. 572. 2·62 in., 0·93 in., 1·79 in.
Ex. 573. 2·36 in., 1·38 in., 1·69 in.
Ex. 574. $\frac{1}{36}, \frac{1}{63360}, \frac{1}{102}, \frac{1}{95040}, \frac{1}{848\cdot\dot{5}7142\dot{8}}, \frac{1}{17\cdot117\dot{8}\ldots}$.
Ex. 575. 3·696 in.
Ex. 576. $\frac{1}{2382336}$.
Ex. 577. 10·00405 . . . miles.

Ex. 578. 1·65 in. nearly.
Ex. 580. 19 miles, 22 miles.
Ex. 581. 2·65 miles nearly.
Ex. 582. 38·42 ft. nearly.
Ex. 584. $\frac{1}{146\cdot85}$.
Ex. 594. 3 miles 7 fur.
Ex. 595. 19 miles.
Ex. 596. $B'C$ = 2·21 metres nearly, CD = 2·55 metres nearly.
Ex. 1131. 37 acres 0 roods $30\frac{50}{121}$ poles.
Ex. 1132. 94 acres 1 rood $2\frac{78}{121}$ poles.
Ex. 1133. 27620 sq. yds.
Ex. 1134. 3·772 acres.
Ex. 1135. 5 acres 0 roods 5·84 poles.
Ex. 1136. 75626 sq. yds.
Ex. 1137. 0·6538 acres.
Ex. 1138. 2·05 acres.
Ex. 1139. 1·499 acres nearly.
Ex. 1140. 5·62 acres nearly.

Part III.

Exp. 355. 2·164 cm.
Exp. 356. 1·41 . . . in.
Exp. 359. 2·94 . . . cm.
Exp. 360. 1·58 . . . in.
Exp. 362. 1·8 in.

Part IV.

Ex. 1261. $\frac{25}{18}$.
Ex. 1262. 0·481.
Ex. 1263. $\sqrt{2}$.
Ex. 1264. $\frac{\sqrt{3}}{2}$.
Ex. 1265. 37 in. to 50 in.
Ex. 1266. 17 ft.
Ex. 1267. 8 in., 10 in.
Ex. 1268. 15 cm., 10 cm.

Ex. 1269. 85°, 153°.
Ex. 1270. 3, $-\frac{1}{2}$.
Ex. 1271. Yes. No.
Ex. 1272. $3\frac{3}{4}$ oz.
Ex. 1273. 150.
Ex. 1274. ·135.
Ex. 1275. 9 sq. in.
Ex. 1276. 4 sq. in.
Ex. 1277. 1 shilling.
Ex. 1278. 1 shilling.
Ex. 1279. $7 + 5\sqrt{2}$.
Ex. 1280. 7·5 per cent.

HINTS TO THE SOLUTIONS OF EXERCISES MARKED WITH AN ASTERISK

HINTS TO THE SOLUTIONS OF EXERCISES MARKED WITH AN ASTERISK.

Ex. 18. $\angle ECD = \angle DCA - \angle ACE$. Also $\angle ECD = \angle ECB - \angle BCD$. $\therefore 2\angle ECD = \angle DCA - \angle BCD$.

Ex. 24. Let OC be the bisector of the acute $\angle AOB$, and OD the bisector of the reflex $\angle AOB$. Then $\angle AOC + \angle AOD = \frac{1}{2}$ (acute $\angle AOB$ + reflex $\angle AOB$) $= \frac{1}{2}$ of 4 rt. $\angle$s.

Ex. 30. Produce FE to G. Then $\angle AEF = \angle BEG$ (Th. 3). Also $\angle CEF = \angle DEG$ (Th. 3). But $\angle AEF = \angle CEF$. $\therefore \angle BEG = \angle DEG$.

Ex. 35. At C in AB let, if possible, CD, CE be drawn $\perp$s to AB. Then it will follow that $\angle DCA$ is at the same time equal to and unequal to $\angle ECA$.

Ex. 46. Let EF cut the $\parallel$s AB, CD at E, F as in Fig. 66. Let EG, EH, FG, FH be the bisectors of the 4 int. $\angle$s. Then $\angle GEF = \angle HFE$. $\therefore$ GE is $\parallel$ to HF (Th. 4). Similarly HE is $\parallel$ to GF. Also $\angle GFH =$ a rt. $\angle$ (Ex. 14). $\therefore \angle$s FHE, HEG, EGF are all rt. $\angle$s (Th. 6).

Ex. 47. All the $\angle$s of the 4-sided fig. formed by the bisectors can be shown to be rt. $\angle$s as in Ex. 46, and it follows that its opp. sides are $\parallel$ (Th. 5).

Ex. 48. Prove by Playfair's Axiom.

Ex. 49. Draw a st. line through the vertex $\parallel$ to the base and then use Th. 6 and Th. 1.

Ex. 53. Draw a st. line cutting AB, CD, EF (Fig. 71) and show that alternate $\angle$s are $=$.

Ex. 54. Let AB, CD be 2 non-$\parallel$ st. lines. Let them be cut by one st. line in E, F respy. and by another st. line in G, H respy. Through E, G draw st. lines $\parallel$ to CD. Then these st. lines will be $\parallel$ to one another (Th. 7), and their correspg. $\angle$s with AB will be $=$ (Th. 6).

Ex. 82. Each $\angle$ of the quadl. formed by the bisectors is an $\angle$ of a $\triangle$ whose other 2 $\angle$s are halves of ext. $\angle$s of the given quadl. Now use Ths. 8, 1 and Ex. 66.

Ex. 83. $\angle BDC = 180° - (\frac{1}{2}\angle B + \frac{1}{2}\angle C)$ (Th. 8). But $\frac{1}{2}\angle B + \frac{1}{2}\angle C = 90° - \frac{1}{2}\angle A$ (Th. 8).

Ex. 84. $\angle B + \angle C + \angle B - \angle C = 2\angle B = 126° + 74°$. $\therefore \angle B = 100°$ and $\angle C = 126° - 100° = 26°$. $\therefore \angle A = 180° - 100° - 26° = 54°$ (Th. 8).

Ex. 85. The bisectors include an $\angle = 180° - \frac{1}{2}$ sum of ext. $\angle$s (Th. 8) $= 180° - \frac{1}{2}(360° -$ sum of base $\angle$s) (Th. 1) $= \frac{1}{2}$ sum of base $\angle$s.

Ex. 86. $\angle BDC = \frac{1}{2}(\angle B + \angle C)$ (Ex. 85). But $\frac{1}{2}(\angle B + \angle C) = 90° - \frac{1}{2}\angle A$ (Th. 8).

Ex. 87. If DEF is the $\triangle$ formed, $\angle$ EDF = $\angle$ EBA + $\angle$ DAB (Th. 8, Cor. 3) = $\angle$ EBA + $\angle$ EBC = $\angle$ CBA. Similarly $\angle$ DFE = $\angle$ BAC and $\angle$ FED = $\angle$ ACB.

Ex. 88. Let ABC be a $\triangle$. Produce BC to D and let CE, a trisector of ext. $\angle$ ACD, be $\parallel$ to BG, a trisector of int. $\angle$ B. Let CF be the other trisector of ext. $\angle$ ACD. Then $\frac{1}{3}\angle$ B = $\angle$ GBC = $\angle$ ECD (Th. 6) = $\frac{1}{3}$($\angle$ B + $\angle$ A) (Th. 8, Cor. 3). $\therefore$ $\frac{1}{3}\angle$ B = $\frac{1}{3}\angle$ A. $\therefore$ $\angle$ B = $\angle$ A. $\therefore$ $\angle$ ACF = $\frac{2}{3}\angle$ A.

Ex. 111. Each ext. $\angle$ of a reg. polygon of n sides = $\frac{4}{n}$ rt. $\angle$s (Th. 9). Now $\frac{4}{n}$ rt. $\angle$s = 15° when n = 24 and = 20° when n = 18, but is not = 25° for any integral value of n.

Ex. 112. Each int. $\angle$ of a reg. polygon of n sides = $\frac{2n-4}{n}$ rt. $\angle$s (Ex. 105). Now $\frac{2n-4}{n}$ rt. $\angle$s = 135° when n = 8 and = 140° when n = 9, but is not = 130° for any integral value of n.

Ex. 113. That the reg. figs. may fit together 4 rt. $\angle$s $\div$ $\frac{2n-4}{n}$ rt. $\angle$s must be a whole number where n = number of sides of each fig. (Ex. 105). This is only possible when n = 3 or 4 or 6.

Ex. 114. Each $\angle$ formed by producing alt. sides of the polygon is at the vertex of a $\triangle$ whose base $\angle$s are ext. $\angle$s of the polygon, and the sum of these base $\angle$s = 2 × 4 rt. $\angle$s (Th. 9). Also the sum of all the $\angle$s of these $\triangle$s = $2n$ rt. $\angle$s (Th. 8).

Ex. 151. Let AB, CD be = and $\parallel$ and drawn in the same sense. Join BC, AC, BD. Then $\angle$ ABC = $\angle$ BCD (Th. 6). Hence $\triangle$ ABC $\equiv$ $\triangle$ BCD (Th. 10). Now use Th. 4.

Ex. 152. Join OP, OQ, OR. Then $\triangle$ MOP $\equiv$ $\triangle$ MOQ (Th. 10), and $\triangle$ NOP $\equiv$ $\triangle$ NOR (Th. 10). $\therefore$ $\angle$ POQ + $\angle$ POR = 2$\angle$ AOB = 2 rt. $\angle$s. Now use Th. 2.

Ex. 168. Let BC, terminated at B, C by AB, AC be bisected at D. Let EF, terminated at E, F by AB, AC also pass through D. Let BG, $\parallel$ to CA, meet FE or FE produced in G. Then $\triangle$ BDG $\equiv$ $\triangle$ CDF (Ths. 6, 11). $\therefore$ ED is not = DF.

Ex. 169. Let AD bisect the vert. $\angle$ A of $\triangle$ ABC meeting BC in D, and let CF, $\perp$ to AD, meet AD in E and AB in F. Then $\triangle$ AEF $\equiv$ $\triangle$ AEC (Th. 11). $\therefore$ $\angle$ AFC, that is $\angle$ FBC + $\angle$ BCF (Th. 8, Cor. 3) = $\angle$ ACF, or $\angle$ FBC + $\angle$ ACB = 2$\angle$ ACF. Again $\angle$ FCB = $\angle$ ACB − $\angle$ ACF = $\angle$ ACB − $\frac{1}{2}\angle$ FBC − $\frac{1}{2}\angle$ ACB (just proved) = $\frac{1}{2}$($\angle$ ACB − $\angle$ FBC).

Ex. 196. $\triangle$ BCF $\equiv$ $\triangle$ CBE (Ths. 12, 10). $\therefore$ $\angle$ AFK = $\angle$ AEL and FB = EC. Hence $\triangle$ AFK $\equiv$ $\triangle$ AEL (Th. 10).

Ex. 197. Let ABC, DEF be 2 $\triangle$s having AB = DE, AC = DF, $\angle$ ABC = $\angle$ DEF. If $\angle$ BAC = $\angle$ EDF, then $\angle$ ACB = $\angle$ DFE (Th. 10). If $\angle$ BAC is not = $\angle$ EDF, one of them must be the greater. Suppose $\angle$ BAC > $\angle$ EDF. At D in ED suppose $\angle$ EDG

made $= \angle$ BAC and DG to meet EF produced in G. Then $\triangle$ BAC $\equiv \triangle$ EDG (Th. 11) and $\angle$ ACB $= \angle$ DGE also AC = DG. But AC = DF. $\therefore \angle$ DFG $= \angle$ DGF (Th. 12). $\therefore \angle$s DGF, DFE are supplementary, $\therefore \angle$s ACB, DFE are supplementary.

Ex. 198. Produce AB to D so that BD = AB. Join CD. Then $\triangle$ ABC $\equiv \triangle$ DBC (Th. 10). Hence ADC is an equilat. $\triangle$ and $\angle$ ACD $= 2 \angle$ ACB.

Ex. 199. $\angle$ C′A′C $= \angle$ A′C′B $+ \angle$ C′BA′ (Th. 8, Cor. 3). But $\angle$ C′A′B′ $= \angle$ ABC. $\therefore \angle$ A′C′B $= \angle$ B′A′C. Hence $\triangle$ C′BA′ $\equiv$ $\triangle$ A′CB′ (Th. 11). Similarly $\triangle$ C′BA′ $\equiv \triangle$ B′AC′.

Ex. 223. Suppose DF drawn $\parallel$ to AB. Let DE cut BC in G. Then $\angle$ DFC $= \angle$ ABC (Th. 6) $= \angle$ ACB (Th. 12). $\therefore$ DF = DC (Th. 13) = BE. Hence $\triangle$ DGF $\equiv \triangle$ EGB (Ths. 3, 6, 11).

Ex. 224. There are 2 converses. (i) If AB = AC and $\angle$ BAD $= \angle$ CAE then AD = AE. (ii) If AB = AC and AD = AE then $\angle$ BAD $= \angle$ CAE. Proof of (i): $\angle$ ABC $= \angle$ ACB (Th. 12). Hence $\triangle$ ABD $\equiv \triangle$ ACE (Th 11). Proof of (ii): $\angle$ ABC $= \angle$ ACB (Th. 12) and $\angle$ ADE $= \angle$ AED (Th. 12). Hence $\triangle$ ABD $\equiv \triangle$ ACE (Th. 11).

Ex. 225. Produce AD and BC to meet (if possible) at E. Then AE = BE (Th. 13) and DE = CE (Th. 13). $\therefore$ AD = BC. If AD and BC do not meet when produced, the $\angle$s of the quadl. are all rt. $\angle$s (Th. 6). $\therefore$ the opp. sides of the quadl. are $\parallel$ (Th. 5). $\therefore$ AD = BC (Ex. 163).

Ex. 226. Produce AD and BC to meet (if possible) at E. Then $\angle$ EDC $= \angle$ ECD (Th. 6) and AE = BE (Th. 13) and DE = CE (Th. 13). $\therefore$ AD = BC. If AD and BC do not meet when produced, the opp. sides of the quadl. are $\parallel$ and therefore = (Ex. 163).

Ex. 227. $\angle$ BDC $= \angle$ DBA $+ \angle$ A (Th. 8, Cor. 3) $= 2 \angle$ A $= \angle$ C. $\therefore$ BC = BD (Th. 13). But BD = AD (Th. 13).

Ex. 228. Join AD. Produce AD to E so that DE = AD. Join EC. Then $\triangle$ CDE $\equiv \triangle$ BDA (Ths. 3, 10). $\therefore$ CE = BA and $\angle$ CED $= \angle$ BAD $= \angle$ CAD. Hence AC = EC (Th. 13) = AB.

Ex. 229. If YXZ is drawn in the same sense as AB, $\angle$ XDZ $= \angle$ BDZ $= \angle$ XZD (Th. 6). $\therefore$ XZ = XD (Th. 13). Simily. XY = XD.

Ex. 230. Let ABC be the $\triangle$ and let $\angle$ B $= 2 \angle$ C. Let D be the foot of the $\perp$ from A on BC. From DC cut off DE = DB. Join AE. Then $\triangle$ ABD $\equiv \triangle$ AED (Th. 10). $\therefore \angle$ AEB $= \angle$ B $= 2 \angle$ C. $\therefore$ $\angle$ EAC $= \angle$ C (Th. 8, Cor. 3) and EC = EA (Th. 13) = BA.

Ex. 231. FE $= \frac{1}{2}$ AB = FD (Ex. 212).

Ex. 232. DF $= \frac{1}{2}$ BC = DE (Ex. 212). Now use Th. 12 and Ex. 211.

Ex. 260. Let ABC be a $\triangle$ and let the $\perp$s at E, F, the mid. pts. of BC, AC, meet at D. Let G be mid. pt. of AB. Join DG, DA, DB, DC. Then $\triangle$ DEB $\equiv \triangle$ DEC (Th. 10). $\therefore$ DB = DC. Simily. DA = DC. $\therefore$ DB = DA. Hence $\triangle$ DGB $\equiv \triangle$ DGA (Th. 14). $\therefore$ $\angle$ DGB $= \angle$ DGA.

Ex. 261. If possible let one $\odot$ cut another in 3 pts. A, B, C. Let P, Q be the centres of the $\odot$s. Join PQ, PA, PB, PC, QA, QB, QC. Then $\triangle$ PAQ $\equiv \triangle$ PCQ (Th. 14) and $\triangle$ PBQ $\equiv \triangle$ PCQ (Th. 14). $\therefore \angle$ QPA $= \angle$ QPB $= \angle$ QPC, which is impossible.

Ex. 262. Let ABC be a △. Through A, B, C suppose EF, FD, DE drawn ∥ to BC, CA, AB respy. Then A, B, C are the mid. pts. of EF, FD, DE (Ex. 165). Now the ⊥s at A, B, C to the sides of the △ DEF are concurrent (Ex. 260) and these are the ⊥s from the vertices to the opp. sides of △ ABC (Th. 6).

Ex. 263. Let K, L be the mid. pts. of AB, AC. Now △ ABD ≡ △ ACD (Th. 14). Hence AE is ⊥ to BC and △ ADG ≡ △ EDG (Th. 10). ∴ AG = EG. Simily. AF = EF. Again △ KAE ≡ △ LAE (Th. 10). ∴ ∠ FED = ∠ GED. Hence EF = EG (Th. 11).

Ex. 264.* △ YBC ≡ △ XCB (Th. 10). ∴ ∠ OBC = ∠ OCB and OB = OC (Th. 13). Hence △ OAB ≡ △ OAC (Th. 14) and ∠ BAO = ∠ CAO. Now use Ex. 121.

Ex. 272. Let BD, CD, the bisectors of the int. ∠s B and C of △ ABC, meet at D. Join DA. Suppose DE, DF, DG drawn ⊥s to BC, CA, AB. Then △ DCE ≡ △ DCF (Th. 11). ∴ DE = DF. Simily. DE = DG. ∴ DF = DG. Hence △ ADF ≡ △ ADG (Th. 15). ∴ ∠ DAF = ∠ DAG.

Ex. 273. Proof similar to that of Ex. 272.

Ex. 274. Let ABCDE be part of a reg. polygon. Let the bisectors of ∠s B and C meet at O. Join OD. Then △ OCB ≡ △ OCD (Th. 10). ∴ ∠ OBC = ∠ ODC. But ∠ OBC = ½ ∠ ABC and ∠ ABC = ∠ EDC. ∴ ∠ ODC = ½ ∠ EDC. Simily. OE bisects ∠ E, etc.

Ex. 275. In the fig. of Ex. 274, suppose OF drawn ⊥ to BC. Then △ OFB ≡ △ OFC (Th. 11). ∴ BF = CF. Simily. ⊥s at mid. pts. of CD, DE . . . will pass through O.

Ex. 284. Produce BA, CD to meet in E. Then BE = CE (Th. 13) and in the △ ADE the greater side has the greater ∠ opp. to it (Th. 16). Hence ∠ BAD > ∠ ADC.

Ex. 285. Let ABC be a △ having A for its vert. ∠ and AB > AC. Join A to D the mid. pt. of BC. Produce AD to E so that DE = AD. Join EC. Then △ ABD ≡ △ ECD (Ths. 3, 10). ∴ ∠ BAD = ∠ CEA and AB = EC. Hence ∠ EAC > ∠ CEA (Th. 16). ∴ ∠ EAC > ∠ BAD.

Ex. 286. Let ABC be a △ having A for its vert. ∠ and AB > AC. Let AD bisect ∠ A and let E be the mid. pt. of BC. Then AE falls between AB and AD (Ex. 285), that is BD > DC.

Ex. 300. Join DC. Then ∠ DEC > ∠ A (Th. 8, Cor. 4) > ∠ ACB (Ex. 67) > ∠ ECD. ∴ DC > DE (Th. 17). Again ∠ BDC > ∠ A (Th. 8, Cor. 4) > ∠ ABC (Ex. 67). ∴ BC > DC (Th. 17). ∴ BC > DE.

Ex. 301. ∠ ACB < ∠ ABC (Th. 16). Hence ∠ BCD > ∠ CBD (Ths. 1, 8). ∴ BD > CD (Th. 17).

Ex. 302. ∠ DEC > ∠ ECF (Th. 8, Cor. 4). But ∠ ECF = ∠ ACB (Th. 3) = ∠ ABC (Th. 12) > ∠ BDE (Th. 8, Cor. 4). ∴ ∠ DEC > ∠ BDE ∴ AD > AE (Th. 17).

Ex. 303. Join A, B, C to O the centre of the semi-circle. Then ∠ OBC = ∠ OCB (Th. 12). ∴ ∠ ABC > ∠ BCA. ∴ AC > AB (Th. 17). Simily. AC > BC.

Ex. 304. $\angle$ **FAE** = $\angle$ **DAE** > $\angle$ **AEF** (Th. 8, Cor. 4). $\therefore$ **EF** > **AF** (Th. 17).

Ex. 305. Let **ABC** be the rt. $\angle$d $\triangle$ and **AD** the given line. Then $\angle$ **BAD** is > or < $\angle$ **ABD** according as $\angle$ **CAD** is < or > $\angle$ **ACD** (Th. 8). $\therefore$ **AD** is < or > **BD** according as it is > or < **CD** (Th. 17).

Ex. 320. Produce **BO** to meet **AC** in **D**. Then **BA** + **AD** > **BD** (Th. 18). $\therefore$ **BA** + **AC** > **BD** + **DC** > **BO** + **OC** (Th. 18).

Ex. 321. Let **O** be the centre of the ⊙ and let **POM**, **PN** be st. lines drawn to the circumfce. Join **ON**. Then **PO** + **ON** > **PN** (Th. 18). But **ON** = **OM**. $\therefore$ **PM** > **PN**.

Ex. 322. Proof similar to that of Ex. 321.

Ex. 323. Proof similar to that of Ex. 321.

Ex. 324. Proof similar to that of Ex. 321.

Ex. 325. (**AB** + **AC**) + (**BC** + **BA**) + (**CA** + **CB**) > (**OB** + **OC**) + (**OC** + **OA**) + (**OA** + **OB**) (Ex. 320). $\therefore$ **AB** + **BC** + **CA** > **OA** + **OB** + **OC**.

Ex. 326. Produce **AD** to **E** so that **DE** = **AD**. Join **EC**. Then $\triangle$ **ADB** ≡ $\triangle$ **EDC** (Ths. 3, 10). $\therefore$ **EC** = **AB**. But **AC** + **CE** > **AE** (Th. 18). $\therefore$ **AB** + **AC** > 2**AD**.

Ex. 327. This follows immediately from Ex. 326.

Ex. 328. **OB** + **OC** > **BC** (Th. 18). But **BC** = **AC** and **AC** > **AO** (Ex. 294).

Ex. 329. Let **D**, **E**, **F** be any 3 pts. within $\triangle$ **ABC**. Join **DE** and produce **DE** both ways to meet **BC**, **AC** in **G**, **H**. Join **GF**, **DF**, **EF**, **HF**. Produce **GF** to meet **AC** (or **AB**) in **K**. Then **DE** + **EF** + **FD** < **GH** + **HF** + **FG** (Th. 18) < **GH** + **HK** + **KG** (Th. 18) < **AB** + **BC** + **CA** (Th. 18).

Ex. 330. Let **AD**, **BE**, **CF** be the medians of the $\triangle$ **ABC**. Then **AD** > **AB** − **BD**, **AD** > **AC** − **CD**, **BE** > **BC** − **CE**, **BE** > **BA** − **AE**, **CF** > **CA** − **AF**, **CF** > **CB** − **BF** (Th. 18, Cor.). $\therefore$ 2 (**AD** + **BE** + **CF**) > **AB** + **BC** + **CA**.

Ex. 331. From **AB** cut off **AD** = **AC**. Join **DE**. Then $\triangle$ **ADE** ≡ $\triangle$ **ACE** (Th. 10). $\therefore$ **DE** = **CE**. Now **BE** − **ED** < **BD** (Th. 18 Cor.). $\therefore$ **BE** − **ED** < **AB** − **AD**. $\therefore$ **AB** − **AC** > **EB** − **EC**.

Ex. 332. Produce **BA** to **D** so that **AD** = **AC**. Join **DE**. Then $\triangle$ **ADE** ≡ $\triangle$ **ACE** (Th. 10). $\therefore$ **DE** = **CE**. Now **BE** + **ED** > **BD** (Th. 18). $\therefore$ **EB** + **EC** > **AB** + **AC**.

Ex. 338. Let **AC**, **AD** be drawn from **A** to the circumfce. of the ⊙ such that $\angle$ **BAC** < $\angle$ **BAD**. Join **C**, **D** to **O** the centre of the ⊙. Case I. Let **C**, **D** be on the same side of **AB**. Then the pt. **D** lies between **C** and **A**. $\therefore$ $\angle$ **COA** > $\angle$ **DOA**. Hence **AC** > **AD** (Th. 19). This solution assumes an obvious property of the circle. Case II. Let **C**, **D** lie on opposite sides of **AB**, and prove by symmetry.

Ex. 339. Proof similar to that of Ex. 338.

Ex. 340. **BC** = **AD** (Ex. 263). Also $\angle$ **BAD** is acute (Th. 6). $\therefore$ **AC** > **BD** (Th. 19).

Ex. 341. $\angle$ ECB > $\angle$ DBC (Th. 16). ∴ BE > DC (Th. 19).

Ex. 342. $\angle$ ABC > $\angle$ ACB (Th. 16). ∴ $\angle$ BCE > $\angle$ DBC (Th. 1). Hence BE > DC (Th. 19).

Ex. 349. $\angle$ ECB > $\angle$ DBC (Th. 20). ∴ AB > AC (Th. 17).

Ex. 350. $\angle$ BCE > $\angle$ CBD (Th. 20). ∴ $\angle$ ABC > $\angle$ ACB (Th. 1). ∴ AC > AB (Th 17).

Ex. 351. Join PR, QS. Now PR > QS (Th. 19). Hence $\angle$ PQR > $\angle$ QPS (Th. 20).

Ex. 352. DB > DC (Th. 17). ∴ $\angle$ BAD > $\angle$ CAD (Th. 20).

Ex. 366. Let PM be $\perp$ to AB from an outside pt. P. Let PN, PR be drawn to AB such that $\angle$ MPN < $\angle$ MPR. Case I. Let N, R be on the same side of M. Then $\angle$ PNM is acute (Ex. 67). ∴ $\angle$ PNR is obtuse (Th. 1). ∴ $\angle$ PNR > $\angle$ PRN (Ex. 67). ∴ PR > PN (Th. 17). Case II. Let N, R be on opposite sides of M, and prove by symmetry.

Ex. 367. If possible let the ⊙ whose centre is O cut AC in 3 pts. A, B, C. Join OA, OB, OC. Then OA = OB = OC, which is impossible (Ex. 366).

Ex. 398. AD, FE are each = and ∥ to BC (Th. 22). Hence ADEF is a ▱gm (Ex. 151) and C is the pt. of intersection of its 2 diags. (Th 22).

Ex. 399. △ AEG ≡ △ BEC (Ths. 3, 10). Hence GA is ∥ to BC (Th. 4). Simily. FA is ∥ to BC. ∴ G, A, F are collinear (Playfair's Ax.).

Ex. 400. Let AE, BE, CF, DF, the bisectors of the $\angle$s A, B, C, D of a quadl. ABCD, form a rect. EGFH. Then $\angle$ EAB + $\angle$ EBA = 1 rt. $\angle$ (Th. 8). ∴ $\angle$ DAB + $\angle$ CBA = 2 rt. $\angle$s. ∴ AD is ∥ to BC (Th. 5). Simily. AB is ∥ to DC.

Ex. 401. $\angle$ APQ = $\angle$ AQP (Th. 12). Also $\angle$ B = $\angle$ C (Th. 12). ∴ $\angle$ APQ = $\angle$ B (Th. 8). ∴ PQ is ∥ to BC (Th. 5).

Ex. 402. Suppose PQ, PR drawn $\perp$s to AB, AC. Also CS $\perp$ to AB and PT $\perp$ to CS. Then PT is ∥ to AB (Th. 5). ∴ $\angle$ TPC = $\angle$ B = $\angle$ C (Ths. 6, 12). Hence △ TPC ≡ △ RCP (Th. 11). ∴ PR = CT. Also PQ = TS (Th. 22). ∴ PQ + PR = CS = a const. length for all positions of P.

Ex. 403. Through P draw a ∥ to a side of the △ and use the result of Ex. 402.

Ex. 404. Use the result of Ex. 197.

Ex. 405. AEDP is a ▱gm. ∴ EA = DP (Th. 22). Simily. FB = CP. ∴ EF = 2 DC.

Ex. 406. Let ABCD be a ▱gm such that AB = 2 AD. Join E, F, the mid. pts. of AB, CD. Then AEFD is a rhombus (Ex. 151) and so is EBCF. Let the 2 diags. of each rhombus intersect in G, H respy. Then GFHE is the ▱gm enclosed by the lines bisecting $\angle$s A, B, C, D (Ex. 380, Th. 4). And EF = AD (Th. 22).

Ex. 407. Complete the ▱gm ABCK. Then BD produced will pass through K (Th. 22). Also BK = 2 BD (Th. 22) and △ BCK ≡ △ HBE (Ths. 6, 1 Cor. 2, 10). ∴ EH = BK = 2 BD.

Ex. 408. Let EG, FH be the extreme $\perp$s from the angular pts. of the squares descd. upon the sides AC, BC of a rt. $\angle$d $\triangle$ ABC upon the hypot. produced. Suppose CD drawn $\perp$ to AB. Then $\triangle$ DBC $\equiv \triangle$ HFB (Ths. 8, 1, 11). $\therefore$ HF = BD. Simily. GE = AD.

Ex. 409. $\angle$ AFB + $\angle$ FBA + $\angle$ BAF = 2 rt. $\angle$s (Th. 8). But $\angle$ FBA + $\angle$ BAF = 4 rt. $\angle$s − 2 × 60° − 1 rt. $\angle$ (Ths. 1, 8) = 150°. $\therefore \angle$ AFB = 30°. Now $\angle$ FEC = 60°. $\therefore$ EC is $\perp$ to DF (Th. 8). Simily. DC is $\perp$ to EF. $\therefore$ FC is $\perp$ to DE (Ex. 262).

Ex. 414. Let D, E be the mid. pts. of the sides AB, AC of a $\triangle$ ABC. Join DE and suppose EF drawn $\parallel$ to AB. Then DE is $\parallel$ to BC (Th. 23, Cor. 2). $\therefore$ DE = BF (Th. 22). Also F is the mid. pt. of BC (Th. 23, Cor. 1).

Ex. 415. Let G be the mid. pt. of AB and suppose AE, GH, BF drawn $\perp$s to CD. Join AF meeting GH in K. Now AE, GH, BF are $\parallel$ (Th. 5). $\therefore$ EH = HF (Th. 23). $\therefore$ HK = $\frac{1}{2}$ AE (Th. 23, Cor. 1, Ex. 414). Simily. GK = $\frac{1}{2}$ BF.

Ex. 416. PR and QS bisect each other (Th. 22). Now use the result of Ex. 415.

Ex. 417. Each line is $\parallel$ to a diag. of the quadl. (Th. 23, Cor. 2).

Ex. 418. These lines are the diags. of a $\square$gm (Ex. 417).

Ex. 419. Each line is $\parallel$ to one of two opp. sides of the quadl. (Th. 23, Cor. 2).

Ex. 420. This follows from Ex. 418, Ex. 419 and Th. 22.

Ex. 421. If the diags. are at rt. $\angle$s, the lines bisecting opp. sides are the diags. of a rect. (Ex. 417). If the diags. are =, the lines bisecting opp. sides are the diags. of a rhombus (Ex. 414).

Ex. 422. DF and EB are = and $\parallel$ (Th. 22). $\therefore$ BF is $\parallel$ to ED (Ex. 151) and they trisect AC (Th. 23, Cor. 1).

Ex. 423. Complete the rect. of which the $\triangle$ ABC is half. BC is one of the diags. of this rect. and the median of the $\triangle$ ABC through A is $\frac{1}{2}$ the other.

Ex. 424. Let E, F be the mid. pts. of the diags. BD, AC of a trapezium ABCD whose $\parallel$ sides are AB, CD. Now the line through E $\parallel$ to DC will pass through the mid. pt. of BC (Th. 23, Cor. 1). But the line through the mid. pt. of BC $\parallel$ to AB will pass through F (Th. 23, Cor. 1).

Ex. 425. Join E, F, the mid. pts. of the non-$\parallel$ sides AD, BC of a trapezium ABCD. Let EF cut the diag. AC in G. Then G is the mid. pt. of AC (see solution of Ex. 424). And EG = $\frac{1}{2}$ one of the $\parallel$ sides of the trapezium and GF = $\frac{1}{2}$ the other (Ex. 414).

Ex. 426. Let G be the pt. of intersection of the medians BE, CF of a $\triangle$ ABC. Join AG. Produce AG to H making GH = AG. Join BH, CH. Then GE is $\parallel$ to HC (Th. 23, Cor. 2), and FG is $\parallel$ to BH (Th. 23, Cor. 2). $\therefore$ GBHC is a $\square$gm and AH passes through the mid. pt. of BC (Th. 22).

Ex. 427. In the fig. of Ex. 426, let D be the mid. pt. of BC, then GD = $\frac{1}{2}$ GH (Th. 22) = $\frac{1}{2}$ AG = $\frac{1}{3}$ AD. Simily. GE = $\frac{1}{3}$ BE and GF = $\frac{1}{3}$ CF.

Ex. 428. Join AC. Let CP, CQ, CA cut BD in R, S, T. Then S is the centroid of the △ ACD (Th. 22, Ex. 426). ∴ DS = $\frac{2}{3}$ DT (Ex. 427) = $\frac{1}{3}$ DB (Th. 22). Simily. BR = $\frac{1}{3}$ DB.

Ex. 429. In the fig. of Ex. 426 if BE = CF, then BG = CG and GE = GF (Ex. 427). ∴ △ BGF ≡ △ CGE (Ths. 3, 10). ∴ BF = CE. ∴ AB = AC. Simily. if BE = AD, BC = CA.

Ex. 430. Let AD, BE, CF be the medians of a △ ABC and let G be its centroid (Ex. 426). Then BG + CG > BC, CG + AG > CA, AG + BG > AB (Th. 18). ∴ 2(BG + CG + AG) > perim., that is 2 × $\frac{2}{3}$ sum of medians (Ex. 427) > perim.

Ex. 431. Let P, Q be the feet of the ⊥s from B and C, the ends of the base of a △ ABC, upon the bisector of the vert. ∠. Join P, Q to M the mid. pt. of BC. Suppose MN drawn ⊥ to the bisector. Then CQ, MN, BP are ||s (Ths. 4, 5). Also BM = CM (Hyp.). ∴ PN = QN (Th. 23). ∴ △ MNP ≡ △ MNQ (Th. 10).

Ex. 432. Proof same as that of Ex. 431.

Ex. 433. Through P, the mid. pt. of the base BC of a △ ABC, suppose PS drawn || to AB cutting AC in Q and the int. and ext. bisectors of the vert. ∠ A in R, S. Then Q is the mid. pt. of AC (Th. 23, Cor. 1). Also SQ = QA = RQ (Ths. 6, 13). ∴ RS = 2 AQ = AC.

Ex. 434. BFGE is a ▱gm (Th. 23, Cor. 2). ∴ FG = BE (Th. 22). Also EG = BF = $\frac{1}{2}$ BA = DE (Ex. 414). Hence △ AED ≡ △ CEG (Ths. 8, 10). ∴ CG = AD.

Ex. 435. Join C to F the mid. pt. of AB. Now ∠ DAC = ∠ AFC + ∠ FCA (Th. 8, Cor. 3). But ∠ AFC = ∠ FCA (Th. 12). Hence FC is || to AE (Th. 4). ∴ BC = CE (Th. 23, Cor. 1).

Ex. 436. Join FC. Now EFDC is a rect. (Th. 23, Cor. 2). Also △ AEG ≡ △ CEG (Th. 10) and △ AEF ≡ △ CEF (Th. 10). Hence ∠ FAG = ∠ FCG. Simily. ∠ FCH = ∠ FBH. Hence AG is || to BH (Th. 4).

Ex. 444. The reqd. locus is a st. line || to the given st. line and bisecting the ⊥ from the given pt. to the given st. line.

Ex. 445. The reqd. locus is a st. line || to the given st. line, on the side of it remote from the given pt. and at a distance from it = the distance from it of the given pt.

Ex. 446. The reqd. locus is a quarter of a ⊙ descd. with the vertex of the rt. ∠ as centre and with $\frac{1}{2}$ the line of given length as radius.

Ex. 447. Let AB, CD cut at E. Draw ||s to CD on both sides of it and at distances from it = the const. diffce. Let them cut AB in F, G. Through F, G draw ||s to the bisector of ∠ AEC. The reqd. locus will lie along these ||s and along lines drawn in a similar way || to the bisector of ∠ AED.

Ex. 448. Let AB, CD cut at E. Draw ||s to CD, on both sides of it, and at distances from it = the const. sum. Let them cut AB in F, G. Through F, G draw ⊥s to the bisector of ∠ AEC. The reqd. locus will lie along these ⊥s and along lines drawn in a similar way ⊥ to the bisector of ∠ AED.

Ex. 461. Draw ||s on both sides of the 2 given st. lines and at the given distance from them. The reqd. pts. are the intersections of these 4 lines. There is no such pt. when the given lines are || and the distance between them is greater or less than twice the given distance.

Ex. 462. Draw ||s on both sides of the given line and at the given distance from it. The reqd. pts. are the intersections of these ||s and the perp. bisector of the line joining the 2 given pts. There is no such pt. when the bisector is || to the given line and at a distance from it greater or less than the given distance.

Ex. 463. Describe a ⊙ with the given pt. as centre and the given distance as radius. The reqd. pts. are the intersections of this ⊙ and one of the ||s drawn on both sides of the given st. line at the given distance from it. There is no such pt. when the ||s lie outside the ⊙. Ignore for the present the special case in which the given pt. lies on the given line.

Ex. 485. Let **AB** = the sum. Produce **AB** to **C** so that **BC** = the diffce. Bisect **AC** in **D** (Prob. 2). Then **AD** and **DB** are the reqd. lengths.

Ex. 486. Bisect **FG** (Prob. 2). Then the line of bisection is the reqd. line.

Ex. 487. Join **AC, BD.** Bisect **AC, BD** (Prob. 2). Then their lines of bisection will cut at the reqd. pt. **P.**

Ex. 488. See fig. 114.

Ex. 493. Draw 2 st. lines bisecting one another at rt. ∟s, each equal to the given diag. and join their extremities.

Ex. 494. Draw 2 st. lines bisecting one another at rt. ∟s, equal respy. to the two diags. and join their extremities.

Ex. 497. Let **AB** be the given st. line and **C, D** the given pts. Draw **CE** ⊥ to **AB** (Prob. 4). Produce **CE** to **F** so that **EF** = **CE.** Join **DF** cutting **AB** in **G.** Join **CG.**

Ex. 498. Const. similar to that of Ex. 497.

Ex. 499. Let **E** be the given pt. and **AB, CD** the given || st. lines drawn in the same sense. Through **E** draw **EF** ⊥ to **AB** meeting, produced if necessary, **CD** in **G** (Prob. 4). From **FA** cut off **FH** = **EG** and from **GC** (if **E** lies between **AB** and **CD**), or from **GD** (if **E** lies outside **AB** and **CD**), cut off **GK** = **EF.** Join **EH, EK.**

Ex. 505. Draw any st. line **PC** from **P** to **AB.** At **P** in **PC** make an ∟ = the given diffce. (Prob. 5).

Ex. 506. Draw any st. line **PC** from **P** to **AB.** At **P** in **PC** make an ∟ = the supplement of the given sum (Prob. 5).

Ex. 507. Let **ABC** be a △ rt. ∟d at **B.** At **B** in **AB** make ∟ **ABD** = ∟ **A** (Prob. 5).

Ex. 515. Let **ABC** be the given ∟ and **D** the given pt. Draw **DE** || to **CB** (Prob. 6), meeting **AB** in **E.** From **EA** cut off **EF** = **BE.** Join **FD** and produce **FD** to meet **BC** in **G.**

Ex. 516. Bisect $\angle$ B (Prob. 1). Let the bisector **BE** cut **AC** in **E.** Draw **ED** $\parallel$ to **CB** (Prob. 6).

Ex. 517. In **AB** take any pt. **D.** Draw **DE** $\parallel$ to the given line (Prob 6) and of the given length. Draw **EQ** $\parallel$ to **BA** meeting **AC** in **Q** (Prob. 6). Draw **QP** $\parallel$ to **ED** meeting **AB** in **P** (Prob. 6).

Ex. 518. Draw **AG** $\parallel$ to **ED** an arm of $\angle$ **DEF** (Prob. 6) meeting **EF** in **H.** Bisect **EH** in **B** (Prob. 2). Join **AB** and produce **AB** to meet **ED** (or **DE** produced) in **C.**

Ex. 519. Bisect $\angle$s **A** and **C** (Prob. 1). Let their bisectors meet at **F.** Draw **DFE** $\parallel$ to **AC** (Prob. 6).

Ex. 520. Let **A** be the lesser base $\angle$. Bisect int. $\angle$ **A** and ext. $\angle$ **C** (Prob. 1). Let their bisectors meet at **F.** Draw **FED** $\parallel$ to **CA** (Prob. 6).

Ex. 521. Let **ABC** be the given $\triangle$. Bisect $\angle$ **B** (Prob. 1). Let its bisector **BD** cut **AC** in **D.** Draw **DE** $\parallel$ to **CB** and **DF** $\parallel$ to **AB** (Prob. 6).

Ex. 522. Let **AB, BC** be the intersecting st. lines. Bisect $\angle$ **B** (Prob. 1). At any pt. **F** in the bisector **BD** draw **FE** $\perp$ to **BD** and $= \frac{1}{2}$ the given length (Prob. 3). Draw **EG** $\parallel$ to **DB** meeting **AB** in **G** and draw **GH** $\parallel$ to **EF** meeting **BC** in **H** (Prob. 6).

Ex. 523. From **CA** or **CA** produced cut off **CF** = the given length. At **F** in **CF** draw **FG** cutting **CB** in **G** and making $\angle$ **CFG** $= \frac{1}{2} \angle$ **A** (Prob. 5). Draw **GE** $\parallel$ to **AB** cutting **AC** in **E** (Prob. 6). Draw **ED** $\parallel$ to **CB** (Prob. 6).

Ex. 528. Join **BC.** Find a pt. **D** in **BC** or **BC** produced such that **BD** = 2 **CD** (Prob. 7). Join **AD.**

Ex. 553. Let **AB** = the given base. Draw **CD** $\perp$ to **AB** at its mid. pt. **C.** Make **CD** = the given sum. Join **AD.** Make $\angle$ **DAE** = $\angle$ **ADC** (Prob. 5). Join **B** to the intersection of **AE, CD.**

Ex. 554. Let **AB** = the perim. Draw **CD** $\perp$ to **AB** at its mid. pt. **C.** Make **CD** = the given altitude. Join **DA, DB.** Make $\angle$ **ADE** = $\angle$ **DAC** and $\angle$ **BDF** = $\angle$ **DBC** (Prob. 5).

Ex. 555. Let **ABC** be the rt. $\angle$. With centre **B** and any radius desc. a $\odot$ cutting **BA, BC** in **D, E** respy. With centres **D, E** and the same radius desc. $\odot$s cutting the first $\odot$ in **F, G.** Join **BF, BG.**

Ex. 556. Let **AD** = the given altitude. At **A, D** draw **EAF, GDH** $\perp$s to **AD** (Prob. 3). Make $\angle$ **EAB** = one of the base $\angle$s and $\angle$ **FAC** = the other (Prob. 5).

Ex. 557. Let **AB** = the base. Make $\angle$ **BAC** = the given base $\angle$ (Prob. 5). Make **AC** = the given sum. Join **BC.** Make $\angle$ **CBD** = $\angle$ **BCA** (Prob. 5).

Ex. 558. Let **AB** = the greatest side. Make $\angle$ **BAC** = the given adj. $\angle$ (Prob. 5). Make **AC** = the given diffce. Join **BC.** Produce **AC** to **E** and make $\angle$ **CBD** = $\angle$ **BCE** (Prob. 5).

Ex. 559. Let **AB** = the perim. Make $\angle$ **BAC** $= \frac{1}{2}$ one of the base $\angle$s and $\angle$ **ABC** $= \frac{1}{2}$ the other (Prob. 5). Make $\angle$ **ACD** = $\angle$ **A** and $\angle$ **BCE** = $\angle$ **B** (Prob. 5).

Ex. 560. Let **AB** = the given sum. Make ∟ **ABC** = 45° (Ex. 490) With **A** as centre and the given hypot. as radius desc. a ⊙ cutting **BC** in **D, E**. Draw **DF, EG** ⊥s to **AB** (Prob. 4). Join **AD, AE.**

Ex. 561. Let **AB** = the given base. Make ∟ **ABC** = ½ given diffce. of base ∟s (Prob. 5). With **A** as centre and given diffce. of remaing. sides as radius desc. a ⊙ cutting **BC** in **D** and **E**. Let **E** lie between **B** and **D**. Join **AE**. Produce **AE** to **F**. Make ∟ **EBG** = ∟ **BEF** (Prob. 5).

Ex. 562. Const. same as that of Ex. 559.

Ex. 563. Let **P, Q, R** be the given pts. and let **P** lie on **AB** the line of the base. At **A** make ∟ **BAC** = 60° and at **B** make ∟ **ABD** = 60° (Ex. 555). Through **Q, R** draw ∥s to **CA, DB** (Prob. 6).

Ex. 564. Const. an equilat. △ **ABC** (Ex. 530). Produce **AC, BC** to **D, E** so that **CD** = **CE** = **CA**. Join **DE**. Draw **GCF** bisecting ∟ **BCD** (Prob. 1) so that **CF** = **CG** = **AB**. Complete the hexagon.

Ex. 565. Const. a △ **ABC** having ⅔ of each of the given medians for its sides (Probs. 7, 10). Produce **CB** to **D** so that **BD** = **CB**. Bisect **CB** in **E** (Prob. 2). Join **AE** and produce **AE** to **F** so that **EF** = **AE**. Join **AD, FD**.

Ex. 566. Let **AB** = the given diag. On each side of **AB** const. an equilat. △ (Ex. 530).

Ex. 567. Let **AB** = the given diffce. Make ∟ **ABC** = ½ rt. ∟ (Ex. 490). Make ∟ **BAC** = ¾ rt. ∟ (Prob. 3, Ex. 491). Then **BC** = side of reqd. square.

Ex. 568. This construction requires a ruler marked in 2 places. Let **ABC** be the given ∟. With **B** as centre and with the distance between the 2 marks on the ruler as radius, desc. a ⊙ cutting **BC, BA** and **CB** produced in **D, E** and **F** respy. Place the ruler so that its edge passes through **E** and its marks lie one on the ⊙ (at **G** say) and the other on **CB** produced (at **H** say). Then ∟ **EHC** = ⅓ ∟ **ABC**. For proof, join **BG** and use Th. 12 and Th. 8, Cor. 3 (Pappus).

Ex. 608. Produce **BC** both ways to meet **FL** in **P** and **KM** in **Q**. Then prove that **CP** and **BQ** are each = the ⊥ from **A** on **BC** (Th. 11).

Ex. 609. **AD** and **BE** are = and ∥ to **CL** (Th. 22). ∴ **ABED** is a ▱gm (Ex. 151). And ▱gm **ABED** = ▱gm **ADLC** + ▱gm **CBEL** (Th. 27, Cor. 2) = ▱gm **AFGC** + ▱gm **CBHK** (Th. 27, Cor. 2).

Ex. 636. Apply one △ to the other so that one pair of = sides coincide and the supplementary ∟s form a straight ∟. Then the other pair of = sides will lie in one and the same st. line (Th. 2). Now use Th. 28, Cor. 2.

Ex. 637. Area of polygon = ½ side of polygon × sum of distances of any pt. within the polygon from the sides (Th. 28, Cor. 1).

Ex. 638. **AFHG** is a ▱gm (Th. 28, Cor. 2). ∴ △ **FGH** = △ **AFG** (Th. 22) = ⅓ △ **ABC** (Th. 28, Cor. 2).

Ex. 639. Produce **AO** to **G** so that **OG** = **AO**. Then **BOCG** is a ▱gm (Th. 28, Cor. 2) and △ **BOG** = △ **ABO** (Th. 28, Cor. 2) = ⅓ △ **ABC** (Ex. 622). But the sides of the △ **BOG** are = in length to **AO, BO, CO** (Th. 22).

Ex. 640. Let **PQRS** be a trapezium having **PQ** $\parallel$ to **RS** and let **T, X** be the mid. pts. of **PQ, RS**. Let **TX** meet **QS** in **V** and join **PV, RV**. Then, by Th. 28, Cor. 2, $\triangle$ **VXS** = $\triangle$ **VXR** and $\triangle$ **VTP** = $\triangle$ **VTQ**. Hence $\triangle$ **PVS** = $\triangle$ **QVR** (Ex. 624) and fig. **PVRS** = $\triangle$ **QSR**. Hence **PVR** is a st. line (Th. 28, Cor. 2).

Ex. 641. **F, G** are pts. of trisection of **AC, AB** (Th. 23). $\therefore$ $\triangle$ **AFH** = $\frac{1}{3}$ $\triangle$ **AHC** (Th. 28, Cor. 2). Similarly $\triangle$ **AGH** = $\frac{1}{3}$ $\triangle$ **AHB** Also $\triangle$ **HDE** = $\frac{1}{3}$ $\triangle$ **HBC** (Th. 28, Cor. 2).

Ex. 642. **VZ** = $\frac{1}{3}$ **PR** (Ex. 428). $\therefore$ $\triangle$ **QVZ** = $\frac{1}{6}$ **PQRS** (Th. 28, Cor. 2). Also **QTSX** = $\frac{1}{2}$ **PQRS** (Th. 28, Cor. 2). $\therefore$ **PQRS** = $3 \times$ **TSXZV**.

Ex. 643. Let **PR, QS** intersect at **O** and complete the sq. **POQV**. Then **PSOV** is a ▱gm and **V, T, S** are collinear (Th. 22). $\therefore$ **PX** = $\frac{1}{3}$ **PQ** (Ex. 428).

Ex. 655. Join **AC, AE, BD, BF**. Then $\triangle$ **AFB** = $\frac{1}{2}$ $\triangle$ **ADB** (Th. 28, Cor. 2). Similarly $\triangle$ **BEA** = $\frac{1}{2}$ $\triangle$ **BCA**. But $\triangle$ **ADB** = $\triangle$ **BCA** (Th. 28, Cor. 2). Hence **EF** is $\parallel$ to **AB** (Th. 29).

Ex. 656. $\triangle$ **PDQ** = $\triangle$ **DPB**. $\therefore$ **PBRD** is a ▱gm (Th. 29). Also **ABCD** is a ▱gm. Hence **CR** = **AP** (Th. 22).

Ex. 669. Through **E** draw **FG** $\parallel$ to **AB** meeting **AD, BC** in **F, G** Then $\triangle$ **EFD** $\equiv$ $\triangle$ **EGC** (Th. 11). $\therefore$ quadl. **ABCD** = ▱gm **ABGF** = 2 $\triangle$ **AEB** (Th. 30).

Ex. 670. **A** is the mid. pt. of **EF** (Th. 23). Hence **EBCF** = 2 $\triangle$ **ABC** (Ex. 669).

Ex. 671. Through the vertices of the quadl. draw $\parallel$s to the diags. Then quadl. = $\frac{1}{2}$ ▱gm so formed (Th. 30) = a $\triangle$ having 2 sides equal to the diags. of the quadl. and the contained $\angle$ equal to that between the diags.

Ex. 672. $\triangle$ **ADF** = $\frac{1}{2}$ ▱gm **ABCD** = $\triangle$ **AED** + $\triangle$ **BEC** (Th. 30). $\therefore$ $\triangle$ **AEF** = $\triangle$ **BEC**. $\therefore$ $\triangle$ **ABF** = $\triangle$ **CFE**.

Ex. 673. $\triangle$ **AFC** + $\triangle$ **BFD** + $\triangle$ **ABF** + $\triangle$ **BFC** = $\triangle$ **ABC** + $\triangle$ **BFD** = $\triangle$ **BCD** + $\triangle$ **BFD**. $\therefore$ $\triangle$ **AFC** + $\triangle$ **BFD** = $\triangle$ **BCD** + $\triangle$ **BFD** − $\triangle$ **ABF** − $\triangle$ **BFC** = $\triangle$ **CFD** − $\triangle$ **ABF**.

Ex. 678. Draw a st. line from **C** $\parallel$ to **AE**, and then use Th. 11.

Ex. 679. $\triangle$ **AEC** = $\frac{1}{2}$ **AE** $\times \perp$ from **C** on **AE** (Th. 28, Cor. 1). Similarly $\triangle$ **AEB** = $\frac{1}{2}$ **AE** $\times \perp$ from **B** on **AE** and $\triangle$ **AED** = $\frac{1}{2}$ **AE** $\times$ $\perp$ from **D** on **AE**. Hence $\triangle$ **AEC** = difference between $\triangle$s **AEB, AED** (Ex. 678) = $\frac{1}{2}$ difference between ▱gms **BE, DE** (Th. 30).

Ex. 680. They have their greatest value when their common vertex is the mid. pt. of the diag. of the given ▱gm. Prove by reductio ad absurdum, using Th. 31 and Th. 27, Cor. 2.

Ex. 694. Draw **CE** $\perp$ to **AB**. Then $CD^2 = DE^2 + CE^2$ (Th. 32). But $DE^2 = 25BE^2$ and $CE^2 = 3BE^2$ (Ex. 690). $\therefore$ $CD^2 = 28BE^2 = 7AB^2$.

Ex. 695. Let **ABCD** be a rect. Then $AB = \frac{1}{\sqrt{2}} \times$ diag. of sq. on **AB** (Ex. 682). Similarly $BC = \frac{1}{\sqrt{2}} \times$ diag. of sq. on **BC**. $\therefore$ rect. **AB, BC** = $\frac{1}{2}$ rect. contained by diags. of sqs. on **AB, BC**.

Ex. 696. Proof similar to that of Ex. 695.

Ex. 697. Through P draw a $\perp$ to two opposite sides of the rect. and use Th. 32.

Ex. 698. $AE^2 + BF^2 + CD^2 = AB^2 - BE^2 + BC^2 - CF^2 + CA^2 - AD^2$ (Th. 32) $= CA^2 - CF^2 + AB^2 - AD^2 + BC^2 - BE^2 = AF^2 + BD^2 + CE^2$ (Th. 32).

Ex. 699. Through A draw a $\perp$ to BC and B′C′; through B a $\perp$ to CA and C′A′; through C a $\perp$ to AB and A′B′. Now use Th. 32 and Ex. 698.

Ex. 700. PR is $\parallel$ to VT (Th. 4). $\therefore \triangle PVT = \triangle RVT$ (Th. 28, Cor. 2). Similarly $\triangle QXS = \triangle RXS$. $\therefore \triangle PVT + \triangle QXS = \frac{1}{2}(\triangle QRT + \triangle PRS) = \frac{1}{2}$ equilat. $\triangle$ descd. on PQ (Ex. 693) = a const.

Ex. 701. Let BE cut CD at F and CA at G. Now $\triangle DAC \equiv \triangle BAE$ (Th. 10). $\therefore \angle FCG = \angle AEG$. Also $\angle FGC = \angle AGE$ (Th. 3). $\therefore \angle CFG = \angle EAG$ (Th. 8) = a rt. $\angle$.

Ex. 702. Draw AP $\perp$ to BE and AQ $\perp$ to CD. Now $\triangle BAE \equiv \triangle DAC$ (Th. 10). $\therefore AP = AQ$ (Ex. 155). Hence $\angle AFE = \angle AFD$ (Th 15).

Ex. 703. Join M to E, a corner of the sq. ACDE, and join N to F, a corner of the sq. ABGF. Then $\triangle BME = \triangle ABC$ (for $\triangle AME + \triangle CMD = \frac{1}{2}$ sq. $AD = \triangle BCD$. $\therefore \triangle AME = \triangle BCM$). Similarly $\triangle CNF = \triangle ABC$. $\therefore \triangle BME = \triangle CNF$. But $BE = CF$. $\therefore AM = AN$ (Th. 29, Cor.).

Ex. 704. Exs. 701, 702.

Ex. 707. Let ABC be a $\triangle$ having $AC^2 > AB^2 + BC^2$. Draw BD $\perp$ to AB and = BC. Join AD. Then $AD^2 = AB^2 + BD^2$ (Th. 32) $= AB^2 + BC^2 < AC^2$. $\therefore AD < AC$. $\therefore \angle ABD < \angle ABC$ (Th. 20).

Ex. 708. Proof similar to that of Ex. 707.

Ex. 709. $n^2 + (\sqrt{2n+1})^2 = (n+1)^2$.

Ex. 710. $4n$ linear units, or $\sqrt{32n^4 + 2}$ linear units.

Ex. 715. Let BC cut EF at G. Join AG, DG. Then $AG = CG$ (Ex. 712). Similarly $BG = DG$. Hence $\angle AGB = \angle CGD$ (Th. 14). $\therefore$ AG and DG are in the same st. line (Ex. 23).

Ex. 716. Fold the fig. about AB. Then semi-$\odot$ ACB coincides with semi-$\odot$ ADB (Th. 34) and AC lies along AD. $\therefore$ C coincides with D and AC = AD.

Ex. 717. The 4 parts coincide with one another when the $\odot$ is folded about the 2 diams. successively (Th. 34).

Ex. 718. The pts. of intersection of the 2 $\odot$s are the images of one another in the line of centres because this line is an axis of symmetry of the fig. (Th. 34, Cor. 2.)

Ex. 719. Join the centres to a pt. of intersection of the $\odot$s and to one another, and see Ex. 718 and Ex. 691.

Ex. 720. The bisector of the vert. $\angle$ can be seen to be an axis of symmetry by folding.

Ex. 721. Let the bisector of one base $\angle$ B cut the bisector of the vert. $\angle$ A at O. Join OC, and show that OC is the bisector of the base $\angle$ C. See Ex. 720.

Ex. 722. See Ex. 720 and Ex. 715.

Ex. 727. Yes, about the mid. pt. of a diag. Prove by Th. 11.

Ex. 728. Let P, Q be the centres of the $\odot$s that intersect at A, O and R the centre of the $\odot$ that intersects the $\odot$ with centre P at B O. Then PBRO and QCRO are rhombi. $\therefore$ PB is $=$ and $\parallel$ to QC (Th. 7). $\therefore$ PQ is $\parallel$ to BC (Ex. 151). But OA is $\perp$ to PQ (Ex. 718) $\therefore$ OA is $\perp$ to BC (Ex. 42). Similarly OB OC are $\perp$s to CA, AB respy.

Ex. 743. Let A, B be the given pts. and C any pt. on AB. Join A, B, C to O, the centre of the $\odot$, and show that OC $<$ OA (Th. 8, Cor. 4, Th. 12, Th. 17).

Ex. 744. Draw $\perp$s from P and Q to AB, and use Th. 35 and Th. 22.

Ex. 745. Draw $\perp$s from the centres to AB, CD, and use Th. 35 and Th. 22.

Ex. 746. Let T be the pt. of intersection and X, V the centres of the $\odot$s. Draw $\perp$s from X, V to PT, QT, RT, ST and $\parallel$s from X, V to PQ, RS. Now proceed as in Ex. 745, and use Ths. 6, 8 and 11.

Ex. 747. Let T be the pt. of intersection and X, V the centres of the $\odot$s. Draw PTQ $\parallel$ to XV and any other st. line RTS, both terminated by the $\odot$s. Draw $\perp$s from X, V to PT, QT, RT, ST and a $\parallel$ from V to RS. Now proceed as in Ex. 745, and use Ths. 6 and 17.

Ex. 748. If PS, QR intersect at the centre of the $\odot$, it follows that $\angle$ QPS $=$ $\angle$ RSP (Ths. 3, 10) and QP is $\parallel$ to SR (Th. 4).

Ex. 754. The pt. can be shown by Th. 14 to lie on the $\perp$ bisector of the chord joining any 2 of the given pts. on the $\odot$.

Ex. 755. The circum-centres are the pts. of intersection of the $\perp$ bisectors of PT, QT, RT, ST (Th. 35, Cor.).

Ex. 756. An isos. trapezium is symmetrical about the $\perp$ bisector of its $\parallel$ sides and a $\odot$ can be descd. through any 3 of its vertices (Th. 36).

Ex. 761. Join E to F, the centre of the $\odot$. Then $\angle$ BFE $=$ $\angle$ FEC (Th. 6) $=$ $\angle$ FCE (Th. 12) $=$ $\angle$ BFD (Th. 6). $\therefore$ arc EB $=$ arc BD (Th. 37).

Ex. 762. Join E and G to K, the centre of the semi-$\odot$. Then $\angle$ DKE $=$ $\angle$ KEH $+$ $\angle$ KHE (Th. 8. Cor. 3) $=$ $\angle$ KGE (Th. 12) $+$ $\angle$ KHE $=$ 3 $\angle$ GKF (Th. 8, Cor. 3, Th. 12). $\therefore$ arc FG $=$ $\frac{1}{3}$ arc DE (Th. 37).

Ex. 767. Let AB, BC, CD be consecutive sides of a reg. polygon inscd. in a $\odot$. Join AC, BD. Then $\triangle$ ABC $\equiv$ $\triangle$ BCD (Ex. 766, Th. 14). $\therefore$ $\angle$ B $=$ $\angle$ C. Similarly remaining $\angle$s of polygon can be shown to be $=$.

Ex. 768. Let DE cut AB at G and AC at H and join D, E to F, the centre of the $\odot$. Then DF, EF are $\perp$s to AB, AC (Ths. 38, 14, 10). Now $\angle$ AGD $=$ $\angle$ EDF $+$ a rt. $\angle$ (Th. 8, Cor. 3) $=$ $\angle$ DEF (Th. 12) $+$ a rt. $\angle$ $=$ $\angle$ AHE (Th. 8, Cor. 3).

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Ex. 785. The sum of the $\perp$s from **R, S** upon **PQ** = 2 × $\perp$ from the centre of the ⊙ upon **PQ** (Ex. 415) = a const. for all positions of **RS.**

Ex. 786. Proof similar to that of Ex. 785.

Ex. 787. See Ex. 780 and Ex. 754.

Ex. 788. Join **F**, the centre of the ⊙, to **B** and **C**. Then △ **ABF** ≡ △ **ACF** (Th. 14). ∴ **BD, CE** are equidistant from **F** (Ex. 155). ∴ **BD = CE** (Th. 39) and **AD = AE.**

Ex. 789. Let **CD, EF** be chords of a ⊙ bisected at **Q, R** by a fixed chord **AB** whose mid. pt. is **P**. Let **PQ** be < **PR**. Join **O**, the centre of the ⊙, to **P, Q, R**. Then **OQ** < **OR** (Th. 35 and Ex. 366). ∴ **CD** > **EF** (Th. 39).

Ex. 823. ∟ **QPR** = 180° − 2 ∟ **PQR** (Ex. 798, Th. 8) = 180° − 2(90° − ∟ **RQS**) (Th. 40).

Ex. 824. The quadl. so formed is a rhombus (Ex. 790, 805). Now see Ex. 820.

Ex. 825. **S, T** are the mid. pts. of **PQ, PR** (Ths. 40, 35). ∴ **ST** = $\frac{1}{2}$ **QR** (Ex. 414).

Ex. 826. Join **O**, the centre of the ⊙, to **S**. Then ∟ **OSP** = ∟ **OPS** (Th. 12) = ∟ **QPS**. ∴ **OS** is ∥ to **PQ** (Th. 4). But **OS** is $\perp$ to the tangent to the ⊙ at **S** (Th. 40).

Ex. 827. Let the in-⊙ touch **AC** at **E** and **AB** at **F**. Then difference between **AB** and **AC** = difference between **BF** and **CE** (Ex. 798) = difference between **BD** and **CD** (Ex. 798).

Ex. 828. Let **A** be the fixed pt. and **AB, AC, BC** the 3 tangents. Join **I**, the centre of the ⊙, to **B** and **C**. Then ∟ **BIC** = 180° − $\frac{1}{2}$(∟ **B** + ∟ **C**) (Th. 8) = 180° − $\frac{1}{2}$ (180° − ∟**A**) (Th. 8) = 90° + $\frac{1}{2}$ ∟ **A** = a const. for all positions of **BC.**

Ex. 829. Join **O**, the centre of the ⊙, to **P** and **Q**. Then ∟ **POQ** = 180° − $\frac{1}{2}$ (∟ **P** + ∟ **Q**) (Th. 8) = 180° − 90° (Th. 6) = 90°.

Ex. 830. The in-centre lies on the bisector of the vert. ∟ (Th. 25), which is also the $\perp$ bisector of the base (Ex. 121) and, therefore, passes through the circum-centre (Th. 24).

Ex. 831. Let **ABC** be the △. Then the internal bisector of ∟ **A** passes through the in-centre **I** (Th. 25) and one of the e-centres, say I_1 (Th. 25), of the △. Also ∟ I_1AI_2 = ∟ I_1AI_3 = 90° (Ex. 14). Similarly I_2B is $\perp$ to I_3I_1.

Ex. 832. (i) $AE_1 = AF_1$ (Ex. 798). $AE_1 + AF_1 = AC + CD_1 + AB + BD_1 = 2s$. ∴ $AE_1 = s$. Similarly $BF_2 = BD_2 = CD_3 = CE_3 = s$.

(ii) **AE = AF** (Ex. 798). **AE + BD + CD = AF + BF + CE** = s. ∴ **AE** = $s - a$. Similarly **BF = BD** = $s - b$; **CD = CE** = $s - c$.

(iii) $BD_1 = BF_1$ (Ex. 798). $BD_1 = CD_1 - BC = s - a$ (i). Similarly $CD_1 = CE_1 = s - a$; $CD_1 = CE_1 = AE_2 = AF_2 = s - b$; $AE_2 = AF_2 = BD_1 = BF_1 = s - c$.

(iv) **AE** = $s - a$ (ii), also $CE_2 = s - a$ (iii). Similarly **BF** = AF_3; **CD** = BD_1.

(v) $EE_1 = FF_1$ (Ex. 801). $EE_1 = AE_1 - AE = s - (s - a)$ (i, ii) $= a$. Similarly $FF_2 = DD_2 = b$; $DD_3 = EE_3 = c$.
(vi) $D_1D_2 = CD_1 + CD_2 = (s - b) + (s - a)$ (iii) $= c$. Similarly $E_2E_3 = a$; $F_3F_1 = b$.
(vii) $DD_1 = BD - BD_1 = (s - b) - (s - c)$ (ii, iii) $= c - b$. Similarly $EE_2 = a - c$; $FF_3 = a - b$.

Ex. 845. The common tangents meet at the in-centre of the △ whose vertices are the centres of the ⊙s (Ths. 41, 40, 15).

Ex. 846. See Ex. 845.

Ex. 847. Let **CD**, the common tangent at **C**, meet **AB** at **D**. Then **CD** is ⊥ to the line of centres (Ths. 41, 40) and **D** is the centre of the ⊙ descd. on **AB** as diam. (Ex. 798). Now see Th. 40, Cor. 2.

Ex. 848. **PR** + **QR** = rad. of ⊙ with centre **P** + rad. of ⊙ with centre **Q** (Th. 41) = const. for all positions of **R**.

Ex. 849. Let **D**, **E** be the centres of ⊙s touching *externally* at **A** and let ∥ tangents be drawn, one to each ⊙, touching the ⊙ with centre **D** at **B** and the ⊙ with centre **E** at **C**. Join **DA**, **DB**, **EA**, **EC**. (1) Suppose the tangents to lie on opp. sides of **A**. Then **BD** is ∥ to **CE** (Th. 40, Ex. 43). ∴ ∠ **BDA** = ∠ **CEA** (Ths. 41, 6). Hence ∠ **DAB** = ∠ **EAC** (Ths. 8, 12). ∴ **AB**, **AC** are in the same st. line (Ex. 23). (2) Suppose the tangents to lie on the same side of **A**. Then **BD** is ∥ to **CE** (Th. 40, Ex. 43). ∴ ∠ **BDA** + ∠ **CEA** = 2 rt. ∠s (Ths. 41, 6). Hence ∠ **BAC** is a rt. ∠ (Ths. 8, 12, Ex. 3). When the ⊙s touch *internally* similar proofs apply.

Ex. 850. Let the chord **RS** be drawn in the ⊙ with centre **P**. Join **PS**, **QT**. Now **PQ** passes through **R** (Th. 41) and ∠ **PRS** + ∠ **QRT** = 90° (Ex. 3). ∴ ∠ **SPQ** + ∠ **TQP** = 180° (Ths. 12, 8). ∴ **PS**, **QT** are = and ∥ (Th. 5). ∴ **ST**, **PQ** are = and ∥ (Ex. 151).

Ex. 851. Let **X**, **V**, **Z** be the centres of the ⊙s and let **XV** pass through **P**, **VZ** through **Q**, **ZX** through **R** (Th. 41). Join **SZ**, **TZ**. Then **SZ**, **TZ** are both ∥ to **XV** (Ex. 841) ∴ **ST** is ∥ to **XV** (Ax. 12).

Ex. 852. Let **D**, **E** be the centres of the ⊙s intersecting at **A**, **B**. Join **AD**, **AE**, **BD**, **BE**, **CD**, **CE**. Now both **AB** and **AC** are ⊥ bisectors of **DE** (Ths. 41, 14, 10).

Ex. 853. Consider 2 positions of the smaller ⊙. In the first let **R** be the pt. of contact. In the second let **S** be the pt. of contact and let **R** now occupy the position **R′** and the centre of the smaller ⊙ the position **Q**. Join **P**, the centre of the larger ⊙, to **R** and **R′** and join **Q** to **R′**. **PQ** produced will pass through **S** (Th. 41). Now ∠ **R′QS** = 2 ∠ **RPS** (Th. 37). But ∠ **R′QS** = 2 ∠ **R′PQ** (Th. 8, Cor. 3, Th. 12). ∴ ∠ **RPS** = ∠ **R′PS**. ∴ **R′** lies on **RP**.

Ex. 872. Join **BC**. Then ∠ **AOC** + ∠ **BOD** = 2 (∠ **EBC** + ∠ **BCE**) (Th. 42) = 2 ∠ **AEC** (Th. 8, Cor. 3).

Ex. 873. Let **I** be the in-centre of △ **ABC**. Join **ID**, **IE**, **IF**. Then ∠ **FDE** = ½ ∠ **FIE** (Th. 42) = ½ (180° − ∠ **A**) (Ths. 40, 8). Similarly ∠ **DEF** = $90° - \frac{B}{2}$, ∠ **EFD** = $90° - \frac{C}{2}$.

Ex. 874. Join E, F, the centres of the $\odot$s, to A, B. Join AB. Then $\angle AEB = \angle AFB$ (Th. 14). $\therefore \angle ACB = \angle ADB$ (Th. 42). $\therefore BC = BD$ (Th 13)

Ex. 875. Join DB, DC. Now $DE = DF$ (Th. 11) and $DB = DC$ (Ex 857). $\therefore \triangle BDE \equiv \triangle CDF$ (Th. 15). $\therefore BE = CF$. $\therefore AE = \frac{1}{2}(AB + AC)$.

Ex. 876. Join AD, AE. Now $\angle AFG = \angle ADF + \angle DAF$ and $\angle AGF = \angle AEG + \angle EAG$ (Th. 8, Cor. 3). But $\angle ADF = \angle EAG$ and $\angle DAF = \angle AEG$ (Ex. 855).

Ex. 877. Join BE, EC. Then BEC is an equilat. $\triangle$. $\therefore \angle BCE = 60°$ (Th. 8). $\therefore \angle BDE = 30°$ (Th. 42).

Ex. 878. PT, QX, RS are the internal bisectors of the $\angle$s of $\triangle PQR$ (Ex. 855). $\therefore$ they are concurrent (Ex. 272).

Ex. 879. Join E, F, the centres of the $\odot$s, to A, B. Join EF. Then $\angle AEB = \angle AFB = 120°$ (Th. 8). $\therefore \angle ACB = \angle ADB = 60°$ (Th. 42). $\therefore \angle CBD = 60°$ (Th. 8). $\therefore CD = DB = BC$ (Ex. 200).

Ex. 880. Let PS lie between PI and PQ. Join SQ. Then $\angle SPI = \angle QPI - \angle QPS = \frac{1}{2}\angle P - \frac{1}{2}(180° - \angle QSP)$ (Ths. 12, 8) $= \frac{1}{2}\angle P - \frac{1}{2}(180° - 2\angle R)$ (Th. 42) $= \frac{1}{2}(\angle R - \angle Q)$ (Th. 8).

Ex. 881. Let AC, PR intersect at E and BD, QS at F. Join PC QD. Now arc AP = arc DS and arc CR = arc BQ (Ex. 861). $\therefore \angle ACP = \angle DQS$ and $\angle CPR = \angle BDQ$ (Ex. 855). $\therefore \angle AEP = \angle BFQ$ (Th. 8, Cor. 3).

Ex. 890. Join PS. Then $\angle SPT = \angle PRQ - \angle PSQ$ (Th. 8, Cor. 3) = const. for all positions of R (Th. 43). $\therefore$ arc ST = const. for all positions of R (Ex. 854).

Ex. 891. $\angle ADE + \angle ADF = \angle ABE + \angle ACF$ (Th. 43) $= \frac{1}{2}(\angle B + \angle C) = \frac{1}{2}(180° - \angle A)$ (Th. 8). Similarly $\angle E = 90° - \dfrac{B}{2}$ and $\angle F = 90° - \dfrac{C}{2}$.

Ex. 892. Join QS. Then RT is $\parallel$ to QS (Th. 23, Cor. 2). $\therefore \angle PTR = \angle PSQ$ (Th. 6) = const. for all positions of S (Th. 43).

Ex. 893. Join SQ. Then $\angle SQR$ and, therefore, $\angle SQT$ is const. (Ex. 889). $\therefore$ arc ST is of const. length (Ex. 854).

Ex. 894. Let the median of $\triangle PTR$ through T meet PR in X and QS in V. Then $\angle QTV = \angle PTX$ (Th. 3) $= \angle TPX$ (Ex. 212) $= \angle QST$ (Th. 43). Hence $\triangle$s QVT, QTS are equiang. (Th. 8) and $\angle QVT = \angle QTS =$ a rt. $\angle$.

Ex. 895. Join BQ, AO, BO. Then $\angle PQB = \frac{1}{2}\angle AOB$ (Th. 42) $= \frac{1}{2}\angle APB$ (Th. 43) $= \frac{1}{2}(\angle PQB + \angle PBQ)$ (Th. 8, Cor. 3). $\therefore \angle PQB = \angle PBQ$. $\therefore PB = PQ$ (Th. 13).

Ex. 896. Join IQ, IR. Then $\angle SRI = \angle SPQ + \angle QRI$ (Th. 43) $= \angle SPR + \angle PRI = \angle SIR$ (Th. 8, Cor. 3). $\therefore SI = SR$ (Th. 13). Similarly $SQ = SI$.

Ex. 897. Produce BO to meet AC at F. Join BE. Then $\triangle$s BOD, AOF are equiang (Ths. 3, 8). $\therefore \angle DBO = \angle OAF = \angle DBE$ (Th. 43). Hence $\triangle DBO \equiv \triangle DBE$ (Th. 11). $\therefore OD = DE$.

Ex. 907. Let ABC be a △ of given base BC and of given vert. ∠ A, and let O be its orthocentre. Then ∠ BOC = 180° − ∠ A (Ths. 3, 8). ∴ ∠ BOC is a const. ∴ locus of O is the arc of a segment containing an ∠ = 180° − ∠ A and having BC for base (Th. 44).

Ex. 908. Let ABC be a △ of given base BC and of given vert. ∠ A, and let I be its in-centre. Then ∠ BIC = 180° − $\frac{1}{2}$ (∠ B + ∠ C) (Th. 8) = 180° − $\frac{1}{2}$ (180° − ∠ A) (Th. 8) = 90° + $\frac{A}{2}$. ∴ ∠ BIC is a const. ∴ locus of I is the arc of a segment containing an ∠ = 90° + $\frac{A}{2}$ and having BC for base (Th. 44).

Ex. 909. Proof similar to that of Ex. 908.

Ex. 910. Let ABC be a △ of given base BC and of given vert. ∠ A, and let the medians AD, BE, CF intersect at the centroid G (Ex. 426). Draw GH ∥ to AB and GK ∥ to AC. Then BH = CK = $\frac{1}{3}$ BC (Ex. 427, Th. 23). ∴ H, K are fixed pts. Also ∠ HGK = ∠ A (Th. 6) = a const. ∴ locus of G is the arc of a segment containing an ∠ = A and having HK for base (Th. 44).

Ex. 911. Let PS, QR intersect at T. Now arc RS is always of const. length (Th. 38). ∴ arc PR + arc QS = a const. ∴ ∠ PQR + ∠ QPS = a const. (Ex. 855). ∴ ∠ PTQ = a const. (Th. 8). ∴ locus of T is the arc of a segment having PQ for base (Th. 44).

Ex. 922. See Ex. 867 and Ex. 912.

Ex. 923. See Ex. 867 and Th. 45.

Ex. 924. See Ex. 922 and Th. 37.

Ex. 925. Let the tangent to the ⊙ at S cut QR at T. Join QS. Then QSR is a rt. ∠d △ (Ths. 45, 1) and TQ = TS (Th. 40, Cor. 2, Ex. 798). ∴ TQ = TR (Ex. 211).

Ex. 926. PQ is a diam. of the ⊙ (Ex. 791). ∴ ∠ PXQ = ∠ PTQ = a rt. ∠ (Th. 45). Hence PTQX is a rect. (Th. 6).

Ex. 927. The locus is (i) the ⊙, (ii) an arc of the ⊙, having for diam. the st. line joining the fixed pt. to the centre of the given ⊙ (Ths. 35, 45).

Ex. 928. Produce RO to meet the ⊙ at T. Join TQ. Then △s PRS, TRQ are equiang. (Ths. 45, 43, 8). ∴ ∠ PRS = ∠ ORQ.

Ex. 929. EF passes through O (Th. 41). Also AC, BD are diams. of the ⊙s (Th. 45). Hence EA is ∥ to CD and FD to BA (Ths. 10, 4). Produce EA and FD to meet at G. Then ∠ EGF is a rt. ∠ (Th. 6) and GF = AB, EG = CD. ∴ $EF^2 = AB^2 + CD^2$ (Th. 32).

Ex. 930. Let PQR be a rt. ∠d △ on the given hypot. QR. Then the locus of P is a ⊙ having QR for diam. (Ex. 912). Produce QP to S making PS = PR. Join SR. Then ∠ QSR = 45° (Th. 8, Cor. 3, Th. 12) and the locus of S is the arc of a segment having QR for base (Th. 44). But QS = QP + PR = a maximum when it is the diam. of the ⊙ QSR (Ex. 888), that is when △ QPR is isos.

Ex. 931. Proof similar to that of Ex. 930.

Ex. 932. See Ex. 931.

Ex. 945. Let the st. line **PQ** be equally inclined to the sides **AD, BC** of a cyclic quadl. **ABCD.** Through **E**, the pt. of intersection of **BA, CD,** draw a st. line ∥ to **PQ** meeting **AD** at **F** and **BC** at **G.** Then ∠ **AFG** = ∠ **BGF** (Th. 6). But ∠ **AFG** (or its supplement) = ∠ **AEF** + ∠ **EAF** (Th. 8, Cor. 3) and ∠ **BGF** (or its supplement) = ∠ **CEG** + ∠ **ECG** (Th. 8, Cor. 3). ∴ ∠ **AEF** = ∠ **CEG** (Ex. 934). ∴ **PQ** is equally inclined to **BA, CD** (Th. 6).

Ex. 946. Let the st. line **PQ** be equally inclined to the sides **AD, BC** of a cyclic quadl. **ABCD.** Let **PQ** meet **AD, BD, AC, BC** at **E, F, G, H** respy. Then ∠ **BFH** = ∠ **BHQ** − ∠ **FBH** (Th. 8, Cor. 3) and ∠ **AGE** = ∠ **AEP** − ∠ **GAE** (Th. 8, Cor. 3). But ∠ **BHQ** = ∠ **AEP** and ∠ **FBH** = ∠ **GAE** (Th. 43).

Ex. 947. See Ex. 946 and Th. 11.

Ex. 948. Let **AD**, the ext. bisector of the vert. ∠ **A** of a △ **ABC**, meet the circum-⊙ at **D.** Join **DB, DC.** Then ∠ **DBC** = ∠ **DCB** (Ex. 934, Th. 43). ∴ **DB** = **DC** (Th. 13).

Ex. 949. Let **D** be a pt. on the circum-⊙ of a △ **ABC** equidistant from the ends of the base **BC.** Then ∠ **DBC** = ∠ **DCB** (Th. 12). Now apply Th. 43 and Ex. 934.

Ex. 950. Let **P** be the pt. of intersection through which **AB, CD** pass and let **Q** be the other. Produce **AC, DB** to meet at **E** and join **PQ, CQ, DQ.** Then ∠ **E** = 180° − (∠ **EAB** + ∠ **EBA**) (Th. 8) = 180° − ∠ **CQD** (Th. 43, Ex. 934) = a const. (Ex. 889).

Ex. 951. Produce **DA** to cut the ⊙ **ACB** again at **E.** Join **BE.** Then **BE** = **BD** (Ex. 874). ∴ ∠ **BDE** = ∠ **BED** (Th. 12) = ∠ **DCA** (Ex. 934). ∴ **AC** = **AD** (Th. 13).

Ex. 952. **A** is the centre of the ⊙ passing through **B, C, D.** ∴ ½ ∠ **BAD** = 180° − ∠ **BCD** (Ths. 42, 46) = ∠ **CBD** + ∠ **CDB** (Th. 8)

Ex. 962. If = chords subtend = ∠s at the circumferences they also subtend = ∠s at the centres (Th. 42). Hence the radii of the ⊙s are = (Ths. 12, 8, 11). If = chords subtend supplementary ∠s at the circumferences they also subtend = ∠s at the circumferences (Th. 46)

Ex. 963. In an acute ∠d △ ∠ **BOC** + ∠ **BAC** = 180° Ths. 3, 8) ∴ circum-⊙s of △ **ABC, BOC** are = (Ex. 962). Similarly circum-⊙s of △s **ABC, COA, AOB** are =. A similar proof applies to *any* △.

Ex. 964. Let the circum-⊙ of △ **BCF** meet **EF** at **G.** Join **CG.** Then ∠ **EDC** + ∠ **EGC** = ∠ **ABC** + ∠ **FBC** (Ex. 934) = 180° (Th. 1). ∴ the circum-⊙ of △ **CDE** passes through **G** (Th. 47).

Ex. 965. Let **T** lie nearer **P** than **R** and let the circum-⊙ of △ **RQT** meet **QS** at **X.** Join **XT, XR.** Then ∠ **TXQ** = ∠ **TRQ** (Th. 43) = ∠ **TPS** (Th. 6). Hence circum-⊙ of △ **PST** passes through **X** (Ths. 1, 47). A similar proof applies when **T** lies nearer **R** than **P.**

Ex. 966. Let **O** be the orthocentre of an acute ∠d △ **ABC.** Then **D, O, E, C** are concyclic (Th. 47). ∴ ∠ **EDC** = ∠ **EOC** (Th. 43) = ∠ **FOB** (Th. 3) = ∠ **FDB** (Ths. 47, 43). Similarly ∠ **DFB** = ∠ **EFA** and ∠ **FEA** = ∠ **DEC.** A similar proof applies to *any* △.

Ex. 967. See Ex. 966.

Ex. 968. The in-centre of $\triangle$ **ABC** is the orthocentre of the $\triangle$ $\mathbf{I_1I_2I_3}$ (Ex. 831).

Ex. 969. Let **P** be a pt. on the minor arc **BC** of the circum-$\odot$ of a $\triangle$ **ABC**, and let **D, E, F** be the feet of the $\perp$s from **P** on **BC, CA** and **AB** produced respy. Join **FD, DE, PB, PC**. Then **FBDP** is a cyclic quadl. (Th. 47). $\therefore$ $\angle$ **PDF** = $\angle$ **PBF** (Th. 43) = $\angle$ **ACP** (Ex 934) = 180° − $\angle$ **PDE** (Ths. 44, 46). $\therefore$ **FD, DE** are in the same st. line. Similar proofs for other cases.

Ex. 970. Let **PQ** pass through **B, QR** through **C, RS** through **D, SP** through **A**. Then $\angle$ **PBA** + $\angle$ **QBC** + $\angle$ **RDC** + $\angle$ **SDA** = 180° (Th. 8). $\therefore$ $\angle$ **B** + $\angle$ **D** = 180° (Th. 1). $\therefore$ **ABCD** is a cyclic quadl. (Th. 47).

Ex. 971. Join **BG, CG, DG** and use Ths. 43, 44, 46, 47.

Ex. 972. Let the circum-$\odot$s of $\triangle$s **AQR, BRP** intersect at **S**. Join **SP, SQ, SR**. Then $\angle$ **RSQ** + $\angle$ **A** = 180° (Th. 46). Similarly $\angle$ **PSR** + $\angle$ **B** = 180°. Hence $\angle$ **QSP** + $\angle$ **C** = 180° (Th. 8). $\therefore$ **S** lies on the circum-$\odot$ of $\triangle$ **CPQ** (Th. 47).

Ex. 973. Let **PQR** be an acute $\angle$d $\triangle$ having $\angle$ **PSR** obtuse, and let the $\perp$s at **Q** and **S** meet at **T**, at **S** and **R** meet at **V**, at **Q** and **R** meet at **X**. Join **PT, PV**. Now **PQTS** is a cyclic quadl. (Th. 47). $\therefore$ $\angle$ **PQS** = $\angle$ **PTS** (Th. 43). Similarly $\angle$ **PRS** = $\angle$ **PVS**. $\therefore$ $\angle$ **QPR** = $\angle$ **TPV** (Th. 8). But $\angle$ **QPR** + $\angle$ **QXR** = 180° (Th. 8) $\therefore$ $\angle$ **TPV** + $\angle$ **TXV** = 180°. $\therefore$ **P, T, X, V** are concyclic (Th. 47). A similar proof applies when $\angle$ **PSR** is acute and for *any* $\triangle$.

Ex. 974. **O, S, R, T** lie on a $\odot$ (Th. 47) whose diam. is **OR** (Th. 45). $\therefore$ $\angle$ **SRT** = 180° − $\angle$ **SOT** (Th. 46) = a const. for all positions of **R**. $\therefore$ **ST** is of const. length for all positions of **R** (Ex. 854, Th. 38).

Ex. 975. Let **P, Q, R** be the mid. pts. of the sides **BC, CA, AB** of an acute $\angle$d $\triangle$ **ABC** whose orthocentre is **O**, and let **X, Y, Z** be the mid. pts. of **OA, OB, OC**. Join **RP, RX, QP, QX**. Then **RX** is ∥ to **BO** (Th. 23, Cor 2). Similarly **RP** is ∥ to **AC**. But **BO** is $\perp$ to **AC**. $\therefore$ **RX** is $\perp$ to **RP**. Similarly **QX** is $\perp$ to **QP**. $\therefore$ **R, P, Q, X** are concyclic (Th. 47). Similarly **Y** and **Z** may be shown to lie on the same $\odot$ as **P, Q, R**. A similar proof applies to *any* $\triangle$.

Ex. 976. In the fig. of Ex. 975 let **D, E, F** be the feet of the $\perp$s from **A, B, C** to **BC, CA, AB**. Now $\angle$s **PDX, PQX** are both rt. $\angle$s. $\therefore$ **P, D, Q, X** are concyclic. Similarly **E** and **F** may be shown to lie on the $\odot$ passing through **P, Q, R, X, Y, Z**.

Ex. 977. Because it passes through the feet of the $\perp$s from the vertices to the opp. sides (Ex. 976).

Ex. 978. Let **O** be the orthocentre and **S** the circumcentre of an acute-angled $\triangle$ **ABC**. Let **D, E, F** be the feet of the $\perp$s from **A, B, C** to **BC, CA, AB**. Let **P, Q, R** be the mid. pts. of **BC, CA, AB**. Let **X, Y, Z** be the mid. pts. of **OA, OB, OC**. Join **SQ, SR, YZ, RQ**. Then **YZ, RQ** are each ∥ to **BC** (Th. 23, Cor. 2) and = ½ **BC** (Ex. 414). $\therefore$ **YZ** is = and ∥ to **RQ**. Hence $\triangle$ **YZO** ≡ $\triangle$ **QRS**

(Th. 11). $\therefore$ SQ = OY = $\frac{1}{2}$ OB. Similarly SP = $\frac{1}{2}$ OA and SR = $\frac{1}{2}$ OC. A similar proof applies to *any* $\triangle$

Ex. 979. In the fig. of Ex. 978 SP = $\frac{1}{2}$ OA (Ex. 978). $\therefore$ SP is = and $\parallel$ to AX. $\therefore$ AS is = and $\parallel$ to XP (Ex. 151). But XP is a diam. of the nine-points $\odot$ (Th. 45) and AS is the circum-radius of $\triangle$ ABC.

Ex. 980. In the fig. of Ex. 978 let OS meet XP at N. Now SP = OX (Ex. 978). Hence $\triangle$ PNS $\equiv$ $\triangle$ XNO (Th. 11). $\therefore$ N is the nine-points centre, and it is midway between O and S.

Ex. 989. Use Th. 43, and see Exp. 333.

Ex. 990. Draw a common tangent to the 2 $\odot$s, and use Ths. 48, 5, 7.

Ex. 991. $\angle$ BAC = $\frac{1}{2}$ $\angle$ BCA (Th. 8) = $\angle$ BPA (Th. 42). Now see Ex. 981.

Ex. 992. Let AF be the tangent at A to the $\odot$ ACE and AG the tangent at A to the $\odot$ ADB. Suppose B to lie within $\angle$ DAF and F within $\angle$ BAG. Now $\angle$ CAE = $\angle$ ABE + $\angle$ AEB (Th. 8, Cor. 3). Also $\angle$ BAF = $\angle$ CEA (Th. 48) and $\angle$ EAG = $\angle$ DBA (Th. 48). $\therefore$ $\angle$ FAG = $\angle$ between BD and EC (Th. 8).

Ex. 993. Let F be the centre of the $\odot$. Then AC = AF = AP (Ex. 212). $\therefore$ ACF is an equilat. $\triangle$. $\therefore$ $\angle$ ABC = 30° (Th. 42), $\therefore$ $\angle$ EDC = 60° (Th. 8). Also $\angle$ DCE = $\angle$ CAF (Ths. 8, 48) = 60°.

Ex. 994. Let O be the centre of the $\odot$. Then $\angle$ AOP = 2 $\angle$ BOP (Th. 37). $\therefore$ $\angle$ ABP = 2 $\angle$ BAP (Th. 42). $\therefore$ $\angle$ BRP + $\angle$ BPR = 2 $\angle$ BAP (Th. 8). But $\angle$ BPR = $\angle$ BAP (Th. 48). $\therefore$ $\angle$ BRP = $\angle$ BAP. Now see Ex. 211.

Ex. 995. Let the $\odot$s touch at F and let their common tangent at F meet AC at G. Then $\angle$ DFG = $\angle$ FAD (Th. 48). Also $\angle$ BFG = $\angle$ FCB (Th. 48). $\therefore$ $\angle$ AFB + $\angle$ CFD = $\angle$ FAD + $\angle$ FCB + $\angle$ AFC = 2 rt. $\angle$s (Th. 8).

Ex. 996. $\angle$ NPR = $\angle$ PQR (Th. 48) = $\angle$ NPB (Ex. 81). Hence $\triangle$ RPN $\equiv$ $\triangle$ BPN (Th. 11).

Ex. 997. $\angle$ CBD = $\angle$ CAD (Th. 43) = $\angle$ CAB (Ex. 798, Th. 14) = $\angle$ between BC and the tangent at C (Th. 48).

Ex. 998. Let PQ, ST be the common tangents and R a pt. of intersection. Then PS is $\parallel$ to QT, for each is $\perp$ to the line of centres. $\therefore$ $\angle$ PRQ = 180° − ($\angle$ RPQ + $\angle$ RQP) (Th. 8) = $\angle$ RPS + $\angle$ RQT (Th. 6) = $\angle$ RST + $\angle$ RTS (Th. 48) = 180° − $\angle$ SRT (Th. 8).

Ex. 1005. On one side of the rect. const. an equiv. $\square$gm having a side AB = one of the given lengths (Ex. 1002). Similarly on AB const. an equiv. $\square$gm having a side = the other given length.

Ex. 1013. On one of the given lengths as base const. a rect. = the given area (Th. 26), then, on the base of this rect., const. an equiv. $\triangle$ having a side = the other given length.

Ex. 1014. Const. a $\square$gm on $\frac{1}{2}$ the base of the given $\triangle$ as base and of the same altitude as the given $\triangle$ but having a side = $\frac{1}{2}$ the sum of the sides of the given $\triangle$.

Ex. 1015. The diags. of a ▱gm bisect each other (Th. 22) and divide the ▱gm into 4 equiv. △s (Th. 28, Cor. 2). Hence const. as in Ex. 1013.

Ex. 1016. Let D be a pt. in the side AB of a △ ABC. Join DC. Draw BE ∥ to DC (Prob. 6) meeting AC produced at E. Join DE.

Ex. 1017. Let ABC be the given △ and let D be a pt. on BC such that BD = the given base. Now const. as in Ex. 1016.

Ex. 1018. Let ABC be the given △ and let D be a pt. on AB (or on BA produced) such that ⊥ from D to BC = the given altitude. Now const. as in Ex. 1016.

Ex. 1019. Const. equiv. △s of the same altitude (Ex. 1018).

Ex. 1020. Const. equiv. △s of the same altitude (Ex. 1018).

Ex. 1021. Const. a △ of the given area having its base along the given st. line and then const. as in Ex. 1018.

Ex. 1022. Join the given pt. to the mid. pt. of a diag. and produce.

Ex. 1023. Const. as in Ex. 1022.

Ex. 1024. Let D be any pt. in the side BC of a △ ABC and let E be the mid. pt. of BC. Join DA. Draw EF ∥ to DA (Prob. 6) meeting a side of the △ at F. Join DF.

Ex. 1025. Trisect the side (Prob. 7) and const. as in Ex. 1024.

Ex. 1026. Divide the side into 4 = parts (Ex. 477) and const. as in Ex. 1024.

Ex. 1027. Divide the side into n = parts (Prob. 7) and const. as in Ex. 1024.

Ex. 1028. Let A be an angular pt. of a quadl. ABCD. Join AC, BD. Through E, the mid. pt. of BD, draw EF ∥ to AC meeting a side of the quadl. at F. Join AF.

Or,

On AB as base const. a △ ABE equiv. to and having ∟ B in common with the quadl. ABCD (Prob. 13). Bisect BE at F (Prob. 2). Join AF. This const. fails when CE > CB.

Ex. 1029. See Ex. 1028, second solution. Trisect BE (Prob. 7).

Ex. 1030. See Ex. 1028, second solution. Divide BE into n = parts (Prob. 7).

Ex. 1031. See Ex. 1027.

Ex. 1032. Produce QP to X, making PX = PS. Join XR. Draw PT ∥ to XR (Prob. 6) meeting QR in T.

Ex. 1033. Find the centroid of the △ (Ex. 426) and join it to the vertices.

Ex. 1034. Let ABCD be the given ▱gm. Produce AB to E so that BE = the given st. line. Complete the ▱gm DAEF. Join FB. Produce FB and DA to meet at G. Complete the ▱gm FDGH. Produce CB to meet GH at K.

Ex. 1035. See Exs. 1012, 1034.

Ex. 1036. Divide the given rectil. fig. into △s by drawing diags. Now see Exs. 1012, 1035.

Ex. 1037. Make a $\triangle$ ABC rt. $\angle$d at B such that AB, BC are respy. = the sides of the given sqs. Then $AC^2 = AB^2 + BC^2$ (Th. 32).

Ex. 1038. See Ex. 1037.

Ex. 1039. Make a $\triangle$ ABC rt. $\angle$d at B such that AB, AC are respy. = the sides of the given sqs. (Prob. 12). Then $BC^2 = AC^2 - AB^2$ (Th. 32).

Ex. 1040. Let AB be the given st. line and CD a side of the given sq. Make $\angle$ ABE = 45° (Ex. 490), and with centre A and rad. CD desc. a $\odot$ cutting BE at F and G. From F or G draw a $\perp$ to AB (Prob. 4).

Ex. 1041. Let ABCD be the given sq. Produce AC to E so that AE = 2 AC and on AE const. a sq.

Ex. 1042. Let ABCD be the given sq. With centre A and rad. AC desc. a $\odot$ cutting AD produced at E. Draw EF $\parallel$ to AB (Prob. 6) meeting AC produced at F.

Ex. 1043. Draw BC $\perp$ to AB (Prob. 3) and = a side of the given sq. Join AC. Make $\angle$ ACD = $\angle$ BAC (Prob. 5) on the same side of AC.

Ex. 1044. Make a $\triangle$ ABC rt. $\angle$d at B such that AB = a side of the given sq. and $\angle$ A = 30° (Ex. 555). Then $BC^2 = \frac{1}{3} AB^2$ (Ex. 690).

Ex. 1045. Let ABCD be the given sq. With centre A and rad. AB desc. a $\odot$ cutting AC at E. Draw EF $\perp$ to BC and EG $\perp$ to CD (Prob. 4). Then FG is a side of the reg. octagon.

Ex. 1046. See Ex. 166 and Th. 32.

Ex. 1047. Draw BC $\perp$ to AB (Prob. 3). Draw AC making $\angle$ BAC = $22\frac{1}{2}$° (Ex. 491) and meeting BC at C. Make $\angle$ ACP = $\angle$ BAC (Prob. 5) on the same side of AC.

Ex. 1048. Let AB = a side of the given sq. Draw BC $\perp$ to AB (Prob. 3). From BC cut off BP = AB, BQ = AP, BR = AQ, BS = AR and so on. Then BQ^2, BR^2, BS^2 . . . shall be twice, three times, four times . . . AB^2 (Th. 32).

Ex. 1049. In the fig. of Ex. 1048 let AB = 1 in. Then BQ, BR, BS . . . shall be $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$ . . . inches.

Ex. 1056. At the given pt. draw the $\perp$ to the diam. through the given pt. (Prob. 3).

Ex. 1057. Draw the shortest chord through the given pt. (Ex. 1056).

Ex. 1058. See Prob. 10.

Ex. 1059. Find S, T the centres of the $\odot$s (Prob. 14). Join R to X, the mid. pt. of ST (Prob. 2). Draw PRQ $\perp$ to RX (Prob. 3).

Ex. 1073. Let P be the given pt. Place a chord in the $\odot$ of the given length and desc. a concent. $\odot$ to touch this chord (Ex. 1060) and draw a tangent to this $\odot$ through P (Prob. 16).

Ex. 1074. Const. as in Ex. 1073.

Ex. 1075. Place a chord in the $\odot$ of the given length and desc. a concent. $\odot$ to touch this chord (Ex. 1060). Now const. as in Ex. 1066.

Ex. 1076. Place a chord in the $\odot$ of the given length and desc. a concent. $\odot$ to touch this chord (Ex. 1060). Now const. as in Ex. 1067.

Exs. 1077, 1078. Solve by the method of the Intersection of Loci.

Ex. 1079. See Th. 25.

Exs. 1080, 1081, 1082, 1083. Solve by the method of the Intersection of Loci.

Ex. 1084. Let a, b, c be the given radii of the 3 $\odot$s. Const. a $\triangle$ having sides $= (a + b), (b + c), (c + a)$ respy. (Prob. 10). Then the vertices of this $\triangle$ will be the centres of the reqd. $\odot$s.

Ex. 1085. Let a, b be the given radii of the 2 $\odot$s that touch externally and c the given radius of the third $\odot$. Const. a $\triangle$ having sides $= (c - a), (c - b), (a + b)$ respy. (Prob. 10). Then the vertices of this $\triangle$ will be the centres of the reqd. $\odot$s.

Exs. 1086, 1087, 1088. Solve by the method of the Intersection of Loci.

Ex. 1089. Let C be the given pt. in the given st. line AB and let D be the centre of the given $\odot$. Draw EDF $\perp$ to AB meeting the circumference of the given $\odot$ at E (Prob. 4) and draw CH $\perp$ to AB (Prob. 3). Join EC cutting the given $\odot$ at G. Join DG and produce DG to meet CH at H. Then H will be the centre of the reqd. $\odot$.

Ex. 1090. Let AB be the given st. line and G the given pt. on the given $\odot$ whose centre is D. Draw EDF $\perp$ to AB meeting the circumference of the given $\odot$ at E (Prob. 4). Join EG and produce EG to meet AB at C. Draw CH $\perp$ to AB (Prob. 3). Join DG and produce DG to meet CH at H. Then H will be the centre of the reqd. $\odot$.

Ex. 1096. Place a chord in each $\odot$ of the length to be intercepted in each case. Desc. concent. $\odot$s to touch these chords (Ex. 1060) and draw a common tangent to these $\odot$s (Prob. 17).

Ex. 1097. In the $\odot$ to be cut by the st. line place a chord of the length to be intercepted. Desc. a concent. $\odot$ to touch this chord (Ex. 1060) and draw a common tangent to this $\odot$ and the $\odot$ to be touched by the st. line (Prob. 17).

Ex. 1104. Draw 3 diams. inclined to one another at angles of 60° (Ex. 555).

Ex. 1105. Insc. a reg. hexagon (Ex. 1104) and join alternate vertices.

Ex. 1106. Insc. a reg. hexagon (Ex. 1104) and bisect the arcs cut off by the sides (Prob. 15).

Ex. 1116. Find the angular pts. of a reg. octagon inscd. in the $\odot$ (Ex. 1103) and draw tangents to the $\odot$ at these pts. (Prob. 16).

Ex. 1117. Find the angular pts. of a reg. hexagon inscd. in the $\odot$ (Ex. 1104) and draw tangents to the $\odot$ at these pts. (Prob. 16).

Ex. 1118. Find the angular pts. of a reg. dodecagon inscd. in the $\odot$ (Ex. 1106) and draw tangents to the $\odot$ at these pts. (Prob. 16).

Ex. 1119. Draw 2 diams. of the $\odot$ inclined to one another at an $\angle$ = the given $\angle$ (Prob. 5) and draw tangents to the $\odot$ at their extremities (Prob. 16).

Ex. 1120. If I_1, I_2, I_3 are the given e-centres, the vertices of the reqd. $\triangle$ are the feet of the $\perp$s from I_1, I_2, I_3 upon I_2I_3, I_3I_1, I_1I_2 respy. (Ex. 968).

Ex. 1127. On AB the given base, describe a segment containing an $\angle$ = the given $\angle$ (Prob. 21). Complete the $\odot$ by describing the arc AEB. Bisect the arc AEB at E (Prob. 15). Join E to the given pt. D in AB. Produce ED to meet the $\odot$ at F. Join AF, BF.

Ex. 1128. Let AB be the given base, α the given vert. $\angle$ and D the given median. Bisect AB at C (Prob. 2). On CB describe a segment containing an $\angle = \angle\ \alpha$ (Prob. 21). With centre A and radius = D describe a $\odot$ cutting the segment at E. Join BE and produce BE to F so that EF = BE. Join AF.

Ex. 1129. Let AB be the given base, C the given vert. $\angle$ and D a line = the sum of the remaining sides. On AB, and on the same side of it, describe 2 segments containing $\angle$s = $\angle$ C and $\frac{1}{2}\angle$ C respy. (Prob. 21). In the second segment place a chord AE = D and cutting the first segment at F. Join BF.

Ex. 1130. Let AB be the given base, C the given vert. $\angle$ and D a line = the difference of the remaining sides. On AB, and on the same side of it, describe 2 segments containing $\angle$s = $\angle$ C and $90° + \frac{1}{2}\angle$ C respy. (Prob. 21). In the second segment place a chord AE = D and produce AE to cut the first segment at F. Join BF.

Ex. 1145. $AH . AL = AH^2 - AH . HL$ (Ex. 1143, ii.) $= AB . BH - AH . HL = HL(AB - AH) = HL^2$.

Ex. 1146. $AC . BD = AB . BD + BC . BD$ (Th. A) $= AB(BC+CD) + BC . BD$ (Th. A) $= AB . CD + BC(AB + BD)$ (Th. A) $= AB . CD + AD . BC$ (Th. A).

Ex. 1147. Let D, E, F be the feet of the $\perp$s from A, B, C to the opp. sides of the $\triangle$. Then $(AP + PD)BC + (BP + PE)CA + (CP + PF)AB = 6\triangle ABC$ (Th. A, Th. 28, Cor. 1). But $PD . BC + PE . CA + PF . AB = 2\triangle ABC$ (Th. 28, Cor. 1).

Ex. 1148. $CE . ED = (AE - AC)(EB + BD) = AE . EB + AE . BD - AC(EB + BD) = AE . EB + AC(AE - EB - BD) = AE . EB + 2AC^2$.

Ex. 1149. Draw AF $\perp$ to BC. Now $BD . CE + BE . CD = BC . DE$ (Ex. 1146) $= 2 . AF . DE = 4\triangle ADE = 2 . AD . AE$.

Ex. 1163. $AC^2 + BD^2 = AB^2 + BC^2 + 2AB . BC$ (Th. B) $+ BD^2 = AB^2 + BD^2 + 2AB(BD - CD) + BC^2 = (AB + BD)^2$ (Th. B) $+ BC^2 - 2AB . CD$.

Ex. 1164. $PS^2 + RQ^2 - 2PQ . RS = PR^2 + RS^2 + 2PR . RS$ (Th. B) $+ RS^2 + SQ^2 + 2RS . SQ$ (Th. B) $- 2PQ . RS = PR^2 + SQ^2 + 2RS(RS + PR + SQ - PQ)$ (Th. A) $= PR^2 + SQ^2$.

Ex. 1165. $BE^2 = AB^2 + AE^2 + 2 . AB . AE$ (Th. B) $= AB^2 + AB . BC + 2AB . AE = AB(AB + BC + 2AE)$ (Th. A) $= AB . DE$.

Ex. 1166. $AF^2 = AB^2 + BF^2 + 2AB . BF$ (Th. B) $= AB^2 + 4AH^2 + 4AH . AB = AB^2 + 4AB . BH + 4AH . AB = AB^2 + 4AB(BH + AH)$ (Th. A) $= 5AB^2$ (Th. A).

Ex. 1172. $(AB + BC)^2 = AB^2 + BC^2 + 2AB.BC$ (Th. B) $= 3AC^2$ (Ex. 1171) $+ 2AC^2$.

Ex. 1181. Join P to the mid. pt. of QR and use Th. D, Cor. 1, or Th. D, Cor. 2, according as S divides QR internally or externally.

Ex. 1182. Use Th. D, Cor. 2.

Ex. 1183. Use Th. D, Cor. 2

Ex. 1184. Use Th. D, Cor. 2.

Ex. 1185. Use Th. D, Cor. 1.

Ex. 1186. Draw a $\perp$ from the rt. $\angle$ to the hypot. and use Th. 23, Th. D, Cor. 2, and Ex. 1158.

Ex. 1187. $AQ.QS + PQ^2 = PS^2$ (Th. D, Cor. 1) $= AR.RS + PR^2$ (Th. D, Cor. 1). But $PQ < PR$. $\therefore PQ^2 < PR^2$. $\therefore AQ.QS > AR.RS$.

Ex. 1188. Let P be the mid. pt. of the side BC of a $\triangle$ ABC and Q the foot of the $\perp$ from A on BC. Suppose $AB > AC$. Then $AB^2 - AC^2 = BQ^2 + QA^2 - CQ^2 - QA^2$ (Th. 32) $= (BQ + QC)(BQ - QC)$ (Th. D) $= 2BC.PQ$.

Ex. 1189. For the foot of the $\perp$ from the vertex to the base is always at a constant distance from the mid. pt. of the base. See Ex. 1188.

Ex. 1190. See Ex. 1188.

Ex. 1191. See Ex. 1188.

Ex. 1192. $AB.BE = AB^2 - AB.AE$ (Ex. 1143, ii.) $= AC^2 - BC^2$ (Th. 32) $- AB.AE = (AC - BC)(AC + BC)$ (Th. D) $- AB.AE = AE(AC + BC - AB) = AE(AE + AB - AB)$.

Ex. 1197. $AQ^2 + QB^2 = 2AP^2 + 2PQ^2$ (Th. F, Cor. 1) $= 2(AQ.QB + PQ^2)$ (Th. D, Cor. 1) $+ 2PQ^2$.

Ex. 1198. Draw AE $\perp$ to BC. Then $BD^2 + CD^2 = 2BE^2 + 2ED^2$ (Th. F, Cor, 2) $= 2AE^2 + 2ED^2 = 2AD^2$ (Th. 32).

Ex. 1199. $EK^2 + GK^2 = 2EH^2 + 2HK^2$ (Th. F, Cor. 2). Also $HF^2 + HG^2 = 2KF^2 + 2HK^2$ (Th. F, Cor. 2). Hence, by subtracting, $EK^2 + GK^2 - HF^2 - HG^2 = 2EH^2 - 2KF^2$. $\therefore EK^2 + 3FK^2 = FH^2 + 3EH^2$.

Ex. 1200. Draw AE $\perp$ to BC. Then $BD^2 + CD^2 = 2BE^2 + 2DE^2$ (Th. F, Cor. 1) $= 3BE^2 - BE^2 + 2DE^2 = AE^2$ (Ex. 690) $+ DE^2 - (BE^2 - DE^2) = AD^2$ (Th. 32) $- BD.CD$ (Th. D).

Ex. 1202. Draw AE $\perp$ to BC. Then $AD^2 = DB^2 + BA^2 + 2.DB.BE$ (Th. 49) $= DB^2 + 3BA^2$. $\therefore 2DB.BE = 2BA^2 = BC^2$ (Th. 32). Hence $DB = BC$.

Ex. 1203. Let the projection of the 5 ft. side on the 3 ft. side measure x ft. Then $7^2 = 3^2 + 5^2 + 6x$ (Th. 49). $\therefore x = \frac{5}{2}$. Hence the 7 ft. side subtends an $\angle$ of 120°.

Ex. 1204. Draw EF $\parallel$ to AB and EG $\perp$ to BC. Then $BE^2 = BF^2 + FE^2 + 2BF.FG$ (Th. 49) $= BF(BF + 2FG)$ (Th. A) $+ CE^2$ (Ths. 6, 13) $= BC.DE + CE^2$.

Ex. 1205. $AB^2 = BC^2 + CA^2 + 2BC.CD$ (Th. 49) $= BD^2 - CD^2 + CA^2$ (Th. B). $\therefore DF^2 + CD^2 = BD^2 + DG^2$. $\therefore FC^2 = GB^2$. (Th. 32).

Ex. 1208. Join OA, OB, OC and use Th. 50.

Ex. 1209. Draw $\perp$s from the ends of one of the $\parallel$ sides to the other and use Ths. 49, 50, A.

Ex. 1210. See Ex. 1209.

Ex. 1211. See Ex. 414, Ex. 417 and Ex. 1210.

Ex. 1212. Draw AE $\perp$ to BC. Now $AB^2 < BC^2 + CA^2$. $\therefore$ AB subtends an acute $\angle$ at C. $\therefore AB^2 = BC^2 + CA^2 - 2BC.CE$ (Th. 50). $\therefore 2CE = CA$. Hence $\angle C = 60°$.

Ex. 1213. $PQ^2 = QR^2 + PR^2 - 2PR.RS$ (Th. 50). Also $PR^2 = QR^2 + PQ^2 - 2PQ.QT$ (Th. 50). $\therefore$, by adding, $2QR^2 = 2PQ.QT + 2PR.RS$.

Ex. 1214. Suppose ABC an acute $\angle$d $\triangle$ and draw HM $\perp$ to FC produced and AL $\perp$ to BC. Then $FH^2 = FC^2 + CH^2 + 2FC.CM$ (Th 49). Also $AB^2 = BC^2 + CA^2 - 2BC.CL$ (Th. 50). But CM = CL (Th. 11). $\therefore FH^2 + AB^2 = 2BC^2 + 2CA^2$. Similarly $KE^2 + BC^2 = 2CA^2 + 2AB^2$ and $DG^2 + CA^2 = 2AB^2 + 2BC^2$. Now add. A similar proof applies to *any* $\triangle$.

Ex. 1215. Suppose AB > AC. Now $AP^2 = PB^2 + AB^2 - 2.BN.AB$ (Th. 50). Also $AP^2 = PC^2 + AC^2 - 2.CM.AC$ (Th. 50). Hence $PC^2 - PB^2 = AB^2 - AC^2$ = a constant for all positions of P. Now see Ex. 1189.

Ex. 1216. Draw AE $\perp$ to BC, and, by using Ths. 49, 50, obtain an expression for $p.AD^2 + q.AD^2$.

Ex. 1223. Let ABCD be the quadl. and P, Q the mid. pts. of AB, CD. Then $AD^2 + AC^2 + BC^2 + BD^2 = 2DQ^2 + 2AQ^2 + 2DQ^2 + 2BQ^2$ (Th. 51) $= CD^2 + 4AP^2 + 4PQ^2$ (Th. 51) $= CD^2 + AB^2 + 4PQ^2$.

Ex. 1224. Let AG produced meet BC at E. Then $DB^2 + DC^2 = 2BE^2 + 2DE^2$ (Th. 51) $= AB^2 + AC^2 - 2AE^2 + 2DE^2$ (Th. 51) $= AB^2 + AC^2 - 10AD^2$ (Ex. 427).

Ex. 1225. Let AC, BD intersect at E. Join PE. Then $2AE^2 + 2PE^2 = 2BE^2 + 2PE^2$ (Th. 51). Hence AE = BE and AC = BD.

Ex. 1226. Let O be the centre of the $\odot$ and E the mid. pt. of CD. Then $AP^2 + BP^2 = 2.OA^2 + 2.OP^2$ (Th. F, Cor. 1) $= 2OC^2 + 2OP^2 = 2CE^2 + 2EP^2$ (Th. 32) $= CP^2 + DP^2$ (Th. 51).

Ex. 1227. Let O be the centre of the $\odot$s, R the mid. pt. of PQ and S the mid. pt. of OT. Then $OP^2 + OQ^2 = 2OR^2 + 2RQ^2$ (Th. 51) $= 2OR^2 + 2RT^2$ (Ex. 212) $= 4OS^2 + 4SR^2$ (Th. 51). But $OP^2 + OQ^2 - 4OS^2$ is a constant. Hence the locus of R is a $\odot$ having S for centre.

Ex. 1232. Produce CI to cut the $\odot$ at A. Then $\angle$ ABO = a rt. $\angle$ (Th. 45) = $\angle$ AQO. $\therefore$ A, Q, B, O are concyclic (Th. 44). $\therefore$ OC.CB = AC.CQ (Th. 52) = 2IC.CQ.

Ex. 1233. AK.KL = BK.KC (Th. 52). But BK.KC cannot be greater than BP^2 (Th. D, Cor. 1) and $BP^2 = OB^2 - OP^2$ (Th. 32).

Ex. 1234. Draw the $\perp$ from B to AC and use Ths. 50, 45, 52. Cor. 1.

Ex. 1235. Draw $\perp$s from the centre of the $\odot$ to the chords and use Ths. B, 52, F, Cor. 1, and 32.

Ex. 1236. $\angle$ BFA = $\angle$ FEA + $\angle$ FAE (Th. 8, Cor. 3) = $\angle$ BCD + $\angle$ FCD (Th. 43) = $\angle$ BCF. $\therefore$ AF touches the circum-$\odot$ of $\triangle$ BCF (Ex. 981). $\therefore AF^2 = AB.AC$ (Th. 52, Cor. 1) = AD.AE (Th. 52).

Ex. 1237. Let R be the foot of the $\perp$ from Q on AB. Then B, P, Q, R are concyclic (Ths. 45, 47). $\therefore$ AP . AQ = AB . AR (Th. 52) = a constant for all positions of P.

Ex. 1238. Let EF cut AB at G. Now DF^2 = DB . DA (Th. 52. Cor. 1) = CA . CB = CE^2 (Th. 52, Cor. 1). $\therefore$ DF = CE. Also $\angle$ DGF = $\angle$ CGE (Th. 3) and $\angle$ DFG is supplementary to $\angle$ CEG. $\therefore$ DG = CG and AG = BG.

Ex. 1239. AE = DF (Ex. 801). $\therefore$ AG = DH (Ex. 1229). $\therefore$ GB . GC = HB . HC (Th. 52, Cor. 1). $\therefore$ GB = HC. $\therefore$ $GH^2 = BC^2 + 4GC . GB$ (Th. E) = $BC^2 + 4GA^2$ (Th. 52, Cor. 1) = $BC^2 + AE^2$.

Ex. 1245. See Th. D.

Ex. 1246. Produce AB to P so that CP = 2AB.

Ex. 1247. Bisect the st. line (Prob. 2).

Ex. 1248. Make the obtuse $\angle$ = 120°.

Ex. 1249. Produce AB to P so that BP = AB $\sqrt{2}$ (Ex. 1049).

Ex. 1250. Let AB be the given st. line and PQ a side of the given square. Bisect AB at C (Prob. 2) and produce AB to D so that $CD^2 = CB^2 + PQ^2$ (Ex. 1037).

Ex. 1251. Let ABC be a $\triangle$ rt. $\angle$d at B. Draw BE $\perp$ to AC (Prob. 4). Bisect CE (Prob. 2).

Ex. 1252. Let AB be the given st. line. Draw BC $\perp$ to AB (Prob. 8) and = $\frac{1}{2}$AB. Join AC. From CA cut off CD = CB and from AB cut off AE = AD. See Ex. 1192.

Ex. 1253. Let AB be the given st. line. Draw BC $\perp$ to AB (Prob. 8) and = $\frac{1}{2}$AB. Join AC. From AC produced cut off CD = CB and from BA produced cut off AE = AD.

Ex. 1254. Let AB be the given st. line. Bisect AB at C (Prob. 2) and produce AB to D so that CD = AC $\sqrt{5}$ (Ex. 1049).

Ex. 1255. Let AB be the given st. line. Divide AB internally at C in medial section (Ex. 1252). On AB desc. a semi-$\odot$ and in it place a chord AD = AC. Join BD.

Ex. 1256. Let A, B be the 2 given pts. and PQ the given st. line, and suppose AB, PQ to meet at C. Along PQ take CD such that CD^2 = AC . CB (Prob. 24). Desc. a $\odot$ through A, B, D (Prob. 18).

Ex. 1257. Let A, B be the 2 given pts. Desc. any $\odot$ through A, B cutting the given $\odot$ at C, D, and suppose AB, CD to meet at E. From E draw a tangent to the given $\odot$ (Prob. 16) and let F be its pt. of contact. Desc. a $\odot$ through A, B, F (Prob. 18).

Ex. 1258. Let AB, CD be the 2 given st. lines and E the given pt. Let AB, CD meet at F, and suppose E to lie within $\angle$ AFC. Draw FG bisecting $\angle$ AFC (Prob. 1) and EG $\perp$ to FG (Prob. 4). Produce EG to H so that GH = EG. Now desc. a $\odot$ through E, H and touching either AB or CD (Ex. 1256).

Ex. 1259. Let A be the given pt. and BC the given st. line. Draw DE a diam. of the given $\odot$ $\perp$ to BC (Prob. 4) and produce DE to meet BC at F. Join DA and produce DA to G so that DA . DG = DE . DF (Ex. 1242). Desc. a $\odot$ through A, G and touching BC (Ex. 1256).

Ex. 1299. Use Th. 28, Cor. 1.

Ex. 1300. Place the $\triangle$s so that $\angle A$ and $\angle P$ may be vertically opposite. Join BQ, CR. Now use Th. 29 and Ex. 1293.

Ex. 1301. Follows from Ex. 1300.

Ex. 1302. $\frac{\triangle PSV}{\triangle QSV} = \frac{\triangle PTV}{\triangle RTV}$ (Ex. 1299 and Th. 54). But $\triangle QSV = \triangle RTV$. Now use Th. 29 and Th. 11.

Ex. 1303. $\frac{BY}{CX} = \frac{AY}{AX}$. Also $\frac{BY}{CX} = \frac{DY}{EX}$. Hence DE is $\parallel$ to XY.

Ex. 1304. Draw CG $\parallel$ to DF, meeting AB or AB produced at G. Then FG = EC.

Ex. 1305. If TS produced cuts PQ at V, then V is the mid. pt. of PQ (Ex. 211).

Ex. 1306. Produce PS and PT to meet QR (produced both ways) at V and X. Then S is the mid. pt. of PV and T is the mid. pt. of PX (Th. 11).

Ex. 1307. Let O be the fixed pt. Join O to the centre of the $\odot$, by a line cutting the circumfce. at Q and R. Let OQ be divided at S and OR at T so that $\frac{OS}{QS} = \frac{OT}{RT} =$ the const. ratio. Join SP, TP. Now $\angle SPT$ is a rt. $\angle$ (Th. 54, Th. 45 (A)), and S, T are fixed pts.

Ex. 1332. $\frac{\triangle AFE}{\triangle DEF} = \frac{AF}{DE}$ (Ex. 1299) $= \frac{FE}{DC}$ (Th. 55) $= \frac{\triangle DEF}{\triangle DEC}$ (Ex. 1299).

Ex. 1333. Let the $\odot$s touch at T and let TX, their common tangent at T, meet PQ at X. If R, S are the centres of the $\odot$s, $\angle RXS$ is a rt. $\angle$ and XT is $\perp$ to RS (Th. 40). $\therefore$ XT is the mean proportional between RT and ST (Ex. 1323). But $XT = \frac{1}{2}PQ$ (Ex. 798).

Ex. 1334. Draw the common tangent at T. $\angle PTQ$ is a rt. $\angle$ (Ex. 798, Ths. 12, 8). $\therefore \angle TPQ = 90° - \angle TQP = \angle QTR$ (Ths. 48, 45).

Ex. 1335. QR subtends a rt. $\angle$ at the centre of the $\odot$ (Ex. 829). Now use Th. 40 and Ex. 1323.

Ex. 1336. Draw EG $\parallel$ to AB, meeting BC at G. Then $\frac{AB}{AC} = \frac{EG}{EC} = \frac{EG}{DB} = \frac{EF}{DF}$.

Ex. 1337. $\angle ADC = 180° - \angle ABC$ (Th. 46) $= 90° + \angle BAE$ (Th. 45) $= \angle AEF$ (Th. 8, Cor. 3).

Ex. 1338. Let ABC, DEF be 2 $\triangle$s having $\angle B = \angle E$ and $\frac{BA}{AC} = \frac{ED}{DF}$. Case I. Suppose $\angle A = \angle D$, then $\angle C = \angle F$ (Th. 8). Case II.

Suppose $\angle A$ not $= \angle D$. At D in ED make $\angle EDF' = \angle A$. Then $\frac{BA}{AC} = \frac{ED}{DF'}$. But $\frac{BA}{AC} = \frac{ED}{DF}$. $\therefore DF = DF'$, and $\angle DFF' = \angle DF'F$. $\therefore \angle DFE + \angle DF'E = 180°$.

Ex. 1339. $\triangle$s PQR, TPQ are equiang., since $\angle Q$ is common and $\angle PRQ = \angle PQR = \angle QPT$. $\therefore \frac{QR}{PQ} = \frac{PQ}{QT}$.

Ex. 1340. Join EC. $\angle BAD = \angle CAE$ and $\angle ABD = \angle AEC$ (Th. 43). $\therefore \triangle$s ABD, AEC are equiang. (Th. 8). $\therefore \frac{AB}{AE} = \frac{AD}{AC}$.

Ex. 1341. $AB \cdot AC = AD \cdot AE$ (Ex. 1340) $= AD \cdot ED + AD^2$ (Th. A, Cor. 2) $= BD \cdot DC + AD^2$ (Th. 52).

Ex. 1342. Produce DA to meet circum-$\odot$ of $\triangle$ ABC at E and proceed as in Exs. 1340, 1341.

Ex. 1343. In $\triangle$ ABC draw AD $\perp$ to BC and let AE be a diam. of circum-$\odot$. Join EC. Then rt. $\angle BDA =$ rt. $\angle ECA$ (Th. 45) and $\angle ABD = \angle AEC$ (Th. 43). $\therefore \triangle$s ABD, AEC are equiang. (Th. 8). $\therefore \frac{AB}{AE} = \frac{AD}{AC}$.

Ex. 1344. Let ABCD be a cyclic quadl. Join AC, BD. Draw BE making $\angle ABE = \angle DBC$ and meeting AC at E. Then $\angle ABD = \angle EBC$ and $\angle BDA = \angle BCE$ (Th. 43). Hence $\frac{BC}{BD} = \frac{CE}{DA}$. $\therefore BD \cdot CE = BC \cdot DA$. Similarly from $\triangle$s ABE, DBC we have $\frac{BA}{BD} = \frac{AE}{DC}$. $\therefore BD \cdot AE = BA \cdot DC$. $\therefore BC \cdot DA + BA \cdot DC = BD \cdot CE + BD \cdot AE = BD \cdot AC$ (Th. A).

Ex. 1345. $PS \cdot QR = PQ \cdot RS + PR \cdot QS$ (Ex. 1344). But $QR = PQ = PR$. $\therefore PS = RS + QS$.

Ex. 1346. Draw PL, SM $\perp$s from P, S to QR, TV. Now $\frac{\triangle PQR}{\triangle STV} = \frac{PL \cdot QR}{SM \cdot TV} = \frac{PL}{SM} \times \frac{QR}{TV} = \frac{PQ}{ST} \times \frac{QR}{TV}$.

Ex. 1347. $\triangle$s PQR, PQS are equiang. (Th. 48). $\therefore \frac{\triangle PQR}{\triangle PQS} = \frac{PQ \cdot QR}{PQ \cdot QS}$ (Ex. 1346).

Ex. 1348. Let TR cut PS at X. Join RP and produce RP to meet QT produced at V. Then $\angle PVT = \angle PQR$ (Ths. 40, 45, Ex. 31) $= \angle VPT$ (Ths. 48, 8). $\therefore TV = TP$ (Th. 18) $= TQ$ (Ex. 798). Now $\frac{TV}{TQ} = \frac{XP}{XS}$ (Ex. 1314).

Ex. 1354. Let T be centre of $\odot$ PQR and V of $\odot$ PQS. Then $\angle PTR = 2\angle RPS = \angle PVS$ (Ths. 42, 48). $\therefore \triangle$s PTR, PVS are similar.

Ex. 1355. Let T be centre of ⊙ PQR and V of ⊙ PQS. Then $\angle$RTQ = $\angle$QVS (Ths. 42, 1). ∴ $\triangle$s RTQ, QVS are similar.

Ex. 1356. See Ex. 1353.

Ex. 1357. See Th. 52, Cor. 2.

Ex. 1358. See Ex. 1353.

Ex. 1359. $\frac{BP}{ER} = \frac{BP + AP}{ER + DR} = \frac{BQ + CQ}{ES + FS} = \frac{BQ}{ES}$. Hence $\triangle$s BPQ ERS are similar and $\frac{PQ}{RS} = \frac{BP}{ER} = \frac{AC}{DF}$.

Ex. 1360. Draw OT ⊥ to PQ and in OT or OT produced take a pt. V such that rect. OT . OV = rect. OR . OS. Join SV. Then $\frac{OT}{OR} = \frac{OS}{OV}$. Hence $\triangle$s OSV, OTR are similar (Th. 57). ∴ $\angle$OSV = $\angle$OTR = a rt. $\angle$. ∴ locus of S is a ⊙ having OV as diam. (Th. 45 (A)).

Ex. 1361. $\triangle$s PAC, PBD are similar (Th. 57). Hence ABCD is a cyclic quadl. (Th. 44). Again $\angle$APB is const. (Th. 43) and $\frac{PC}{PA}$ is a const. ratio. ∴ $\angle$PCA is const. (Th. 54). ∴ $\angle$ACB is const. (Th. 1). ∴ circum-⊙ of quadl. ABCD is a fixed ⊙ and CD subtends a const. $\angle$ in it. ∴ CD is of const. length. ∴ CD is at a const. distance from the centre of a fixed ⊙ (Th. 39 (A)).

Ex. 1373. $\frac{AE}{EC} = \frac{AB}{BC} = \frac{AB^2}{BD^2}$ (Ex. 1282) $= \frac{AD^2}{CD^2}$ (since $\triangle$s ABD, DBC are equiang.).

Ex. 1374. Let A, B be the fixed pts. and P the moving pt. Then $\frac{PA}{PB}$ = the const. ratio. Bisect $\angle$APB internally by PQ meeting AB at Q and externally by PR meeting AB produced at R. Then $\frac{AQ}{BQ} = \frac{AR}{BR}$ = the const. ratio. ∴ Q, R are fixed pts. and $\angle$QPR is a rt. $\angle$.

Ex. 1375. $\angle$TVX = $\angle$SVX (Ex. 855). ∴ $\frac{VS}{VT} = \frac{SX}{TX}$. Similarly $\frac{ZS}{ZT} = \frac{SX}{TX}$.

Ex. 1376. Let BD cut AC at E. Then $\frac{AB}{BC} = \frac{AE}{EC}$. ∴ $\frac{AB + BC}{BC} = \frac{AE + EC}{EC}$. But $\frac{BC}{EC} = \frac{BD}{AD}$ (since $\triangle$s BEC, BAD are similar).

Ex. 1377. Let ⊥ from P meet QR at S and let tangent at P meet QR produced at T. Now $\angle$RPT = $\angle$PQR (Th. 48) = $\angle$RPS (Ex. 81) and $\angle$QPR is a rt. $\angle$ (Th. 45). Now see Exs. 1362, 1365.

Ex. 1378. $\frac{AB}{AC} = \frac{BD}{CD} = \frac{BF}{DE}$ (Th. 55). Also $\frac{AB}{AC} = \frac{BD}{CD} = \frac{DF}{CE}$. ∴ $\frac{AB}{AC} \times \frac{AB}{AC} = \frac{BF}{DE} \times \frac{DF}{CE}$. But DE = DF.

Ex. 1379. Draw the common tangent at P. $\angle QPX = \angle PTX - \angle TRX$ (Th. 48) $= \angle TXR$ (Th. 8, Cor. 3) $= \angle TPX$ (Th. 48). $\therefore \frac{PQ}{PR} = \frac{QX}{RX}$. Again $\angle PST = \angle PQR$ (Th. 48.) $\therefore$ ST is $\parallel$ to QR (Th. 5) and $\frac{PQ}{PR} = \frac{PS}{PT}$ (Th. 54 Cor.).

Ex. 1387. Let BCP, CAQ, ABR be similar $\triangle$s described on the hypot. BC and the sides CA, AB of a rt. $\angle$d $\triangle$ ABC. Then $\frac{\triangle CAQ}{\triangle BCP} = \frac{CA^2}{BC^2}$ and $\frac{\triangle ABR}{\triangle BCP} = \frac{AB^2}{BC^2}$ (Th. 59). $\therefore \frac{\triangle CAQ + \triangle ABR}{\triangle BCP} = \frac{CA^2 + AB^2}{BC^2}$. But $CA^2 + AB^2 = BC^2$ (Th. 82).

Ex. 1388. $\triangle$s RST, PTO are similar (Th. 40). $\therefore \triangle RST : \triangle PTO = TR^2 . TP^2$ (Th. 59). Now use Th. 52 Cor. 1 and Ex. 1282.

Ex. 1389. ACDE is a cyclic quadl. (Th. 44). $\therefore \angle BED = \angle ACB$ (Ex. 984) and $\triangle$s DBE, ABC are similar.

Ex. 1390. Let ABC, DEF be the 2 isos. $\triangle$s and let AG, DH be their altitudes. Then $\frac{AG . BC}{DH . EF} = \frac{BC^2}{EF^2}$ (Th. 28, Cor. 1). $\therefore \frac{AG}{DH} = \frac{BG}{EH}$. Hence $\angle B = \angle E$ (Th. 57).

Ex. 1391. $\triangle$s PTQ, QOR are similar since each is isos. and $\angle QOR = \angle PTQ$ (Th. 47, Ex. 984). $\therefore \frac{\triangle PTQ}{\triangle QOR} = \frac{PQ^2}{QR^2}$ (Th. 59) $= \frac{PS^2}{SQ^2} = \frac{PS}{SR}$ (Exs. 1828, 1282).

Ex. 1392. $\frac{\triangle BPD}{\triangle BPC} = \frac{BD}{BC}$ (Ex. 1299) $= \frac{BD^2}{BA^2}$ (Exs. 1323, 1282) $= \frac{BA^2}{BC^2}$ $= \frac{\triangle ABR}{\triangle BPC}$ (Th. 59). $\therefore \triangle ABR = \triangle BPD$. Similarly $\triangle ACQ = \triangle CPD$.

Ex. 1400. Find the 3rd proportional to (1) 1 in. and a in., (2) a in. and a^2 in., (3) a^2 in. and a^3 in.

Ex. 1401. Divide the given perimeter into segments proportional to the 3 given st. lines (Ex. 1398) and then use Prob. 10. Construction impossible when two of the three given st. lines are not greater than the third.

Ex. 1402. Draw PD $\parallel$ to BA, meeting AC at D (Prob. 6). From DC cut off DR so that $\frac{DR}{DA} = \frac{5}{3}$.

Ex. 1403. The locus of the vertex of a $\triangle$ of given base, and whose sides are proportional to 2 given st. lines is a $\odot$ (Ex. 1374). The locus of the vertex of a $\triangle$ of given base and of one given base $\angle$ is a st. line. The required vertex is the intersection of these loci.

Ex. 1404. See Ex. 1374 and Prob 21.

Ex. 1429. BC must subtend a rt. ∟ at K.

Ex. 1430. The diags. of the ▱gm must be at rt. ∟s to one another.

Ex. 1431. The reqd. st. line is ∥ to the line of centres.

Ex. 1432. The reqd. pt. is the foot of the ⊥ from the centre of the given ⊙ to the given st. line.

Ex. 1433. The reqd. st. line is bisected at P.

Ex. 1434. The reqd. st. line is equally inclined to OX and OY.

Ex. 1435. The reqd. △ is isos.

Ex. 1436. The reqd. pts. are at the intersection of the circumfce. with the ⊥ from the centre to AB.

Ex. 1437. The reqd. position of C is an intersection of the circumfce. with the perp. bisector of AB.

Ex. 1438. See Ex. 1437.

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